14

Finite Element Analysis of Shells

14.1 Introduction

Shells are surface structures, that is, structures for which one dimension, the thickness, is much smaller than the other two dimensions. In the previous Chapter, we saw that the deformation of a beam is quantified based on the configuration of the reference line. In the same fashion, the deformation of a shell is quantified from the configuration of a *reference surface*, as shown in Figure 14.1. The figure also shows the local coordinate system of a flat shell, as well as various forces that can possibly act on the shell. The reference surface lies in the local *xy*-plane, while the direction of the shell's thickness is along the local *z*-axis.

As shown in Figure 14.1, a shell resists forces that lie in the plane of the reference surface by means of its *membrane resistance*, and forces perpendicular to the reference surface by means of its *flexural (plate) resistance*. A shell that is only subjected to perpendicular loading and only develops flexural resistance is also called a *plate*.

Many textbooks are dedicated *to plate theory and analysis, in which the membrane resistance is always neglected*, but this text will not separately examine plates. The major reason is that the concept of a plate is merely a mathematical convenience (if a surface structure has flexural stiffness and resistance, it must also have membrane resistance!). We can always analyze plate problems as a special case of shell problems where there is no membrane loading. We will begin our discussion with the simplest case of a *planar shell*, also called a *flat* shell, for which the shell is a flat planar body.

There are two theories regarding the kinematics of a shell's flexural deformation. The first one is the *Reissner-Mindlin* shell theory, which can be thought of as the extension of Timoshenko beam theory to shells. The basic assumptions of the Reissner-Mindlin theory are the following:

1) Plane sections normal to the undeformed *mid-surface* remain plane, but not normal to the mid-surface, in the deformed configuration.
2) The normal stress along the direction of the thickness is zero, $\sigma_{zz} = 0$.
3) The effect of the change of thickness to the displacements of the shell is neglected, which simply means that when we establish the kinematics of the shell, we will be assuming that the displacement along the local *z*-axis will not be varying along the thickness: $u_z(x, y, z) \approx u_{oz}(x, y)$.

Fundamentals of Finite Element Analysis: Linear Finite Element Analysis, First Edition.
Ioannis Koutromanos, James McClure, and Christopher Roy.

Companion website: www.wiley.com/go/koutromanos/linear

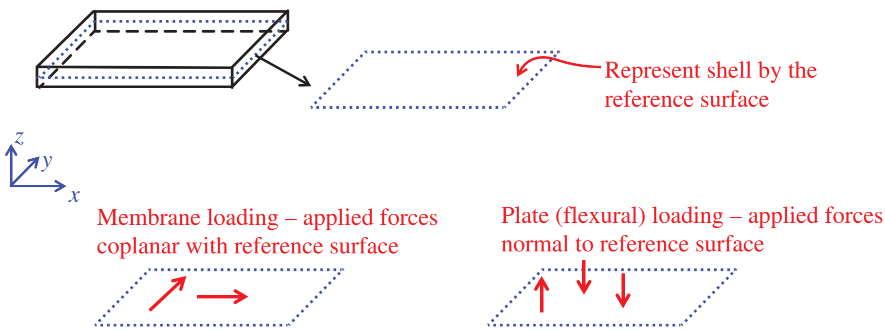


Figure 14.1 Shell reference surface and types of loading.

4) The through-thickness shear stresses of the shell, σ_{yz} and σ_{zx}, are assumed to be constant over the thickness (similar to the assumption for Timoshenko beams in Section 13.8).

The stress components (with example distributions along the thickness d) of a Reissner-Mindlin flat shell are shown in Figure 14.2. The example stress distributions are those corresponding to linear elasticity and homogeneous material throughout the entire thickness of the shell.

The second popular shell theory is the *Kirchhoff-Love* (K-L) theory, which is an extension of the Euler-Bernoulli beam theory (i.e., we do not have through-thickness shear deformations). This theory is *valid for thin shells*, that is, shells wherein the thickness is much smaller than the in-plan dimensions. The finite element analysis of K-L shells requires multidimensional shape functions that satisfy C^1 continuity. It turns out that it is very hard to obtain such shape functions in multiple dimensions. A recently developed method, called *Isogeometric Analysis* (Cottrell et al. 2007) relies on the use of B-splines for shape functions and can resolve this difficulty. If we are to use the standard finite element method for K-L shells, we need to resort to specifically designed *hybrid*

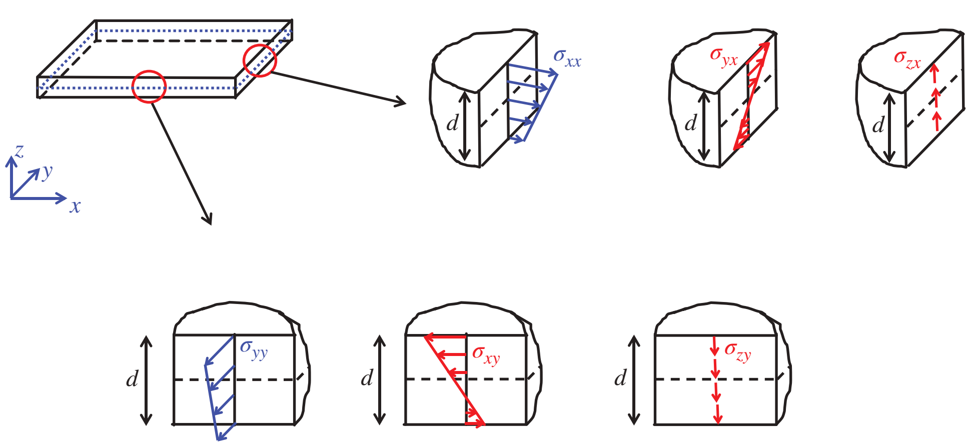


Figure 14.2 Planar Reissner-Mindlin Shell and stress components acting on two of the shell's faces.

finite elements,[1] using the so-called *discrete Kirchhoff quadrilateral (DKQ) or discrete Kirchhoff triangular (DKT)* element formulation. In a discrete Kirchhoff element, the condition for zero through-thickness shear strain is satisfied at a set of discrete points (not over the entire domain). It turns out that this approach leads to convergent finite element solutions for thin shells. The present Chapter is primarily focused on Reissner-Mindlin shell theory. A description of the DKQ element formulation is provided in Section 14.9.

Let us now continue with some further kinematic considerations for shells. The kinematic quantities of the mid-surface are shown in Figure 14.3. We have five quantities, $u_{ox}(x, y)$, $u_{oy}(x, y)$, $u_{oz}(x, y)$, $\theta_1(x, y)$, and $\theta_2(x, y)$, drawn in Figure 14.3 with the directions that will be assumed as positive. The rotation field θ_1 is drawn in the negative y-direction, because we want the deformed shape of the shell for each of the two elevation views, 1-1 and 2-2 in Figure 14.3, to have exactly the same form of deformations as that of two-dimensional Timoshenko beams aligned with the x-axis (elevation 1-1) or y-axis (elevation 2-2).

In the remainder of this chapter, we will be assuming that the reference surface coincides with the *mid-surface* of the shell—that is, a surface that passes through the middle of the shell's thickness at every point. This assumption means that the following mathematical condition will apply.

$$\int_{-d/2}^{d/2} z dz = \left[\frac{z^2}{2}\right]_{-d/2}^{d/2} = \frac{1}{2}\left(\frac{d}{2}\right)^2 - \frac{1}{2}\left(-\frac{d}{2}\right)^2 = 0 \tag{14.1.1}$$

where d is the thickness of the shell.

The three components of the displacement field, $u_x(x, y, z)$, $u_y(x, y, z)$, and $u_z(x, y, z)$, at any location (x, y, z) in a shell, can be determined from the five kinematic quantities of the mid-surface as follows.

$$u_x(x,y,z) = u_{ox}(x,y) - z \cdot \theta_1(x,y) \tag{14.1.2a}$$

$$u_y(x,y,z) = u_{oy}(x,y) - z \cdot \theta_2(x,y) \tag{14.1.2b}$$

$$u_z(x,y,z) \approx u_{oz}(x,y) \tag{14.1.2c}$$

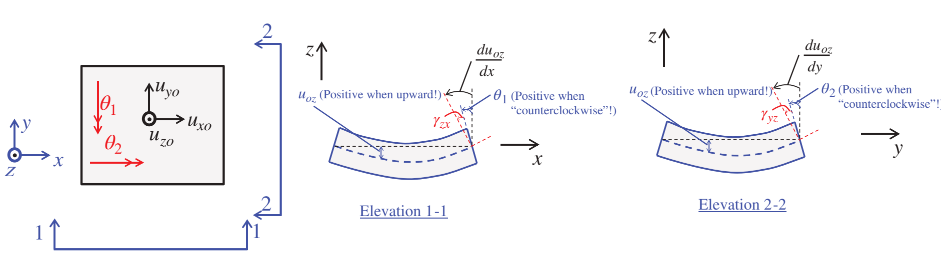


Figure 14.3 Plan view of shell mid-surface, elevation views of deformed shell and definition of kinematic quantities of the mid-surface.

1 In a hybrid finite element, we approximate a specific field in the interior of the element and a different field along the boundary of the element (for example, we may have approximation for the stresses in the interior of an element and for the displacements along the boundary of the element).

We can also define the strain components, through appropriate partial differentiation (as explained in Section 7.2) of the displacement field defined by Equations (14.1.2a–c):

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial u_{ox}}{\partial x} - z \cdot \frac{\partial \theta_1}{\partial x} = \varepsilon_{ox} - z \cdot \varphi_{11} \tag{14.1.3a}$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial u_{oy}}{\partial y} - z \cdot \frac{\partial \theta_2}{\partial y} = \varepsilon_{oy} - z \cdot \varphi_{22} \tag{14.1.3b}$$

$$\gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} = \frac{\partial u_{ox}}{\partial y} + \frac{\partial u_{oy}}{\partial x} - z \cdot \frac{\partial \theta_1}{\partial y} - z \cdot \frac{\partial \theta_2}{\partial x} = \gamma_{oxy} - z \cdot 2\varphi_{12} \tag{14.1.3c}$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \approx 0 \tag{14.1.3d}$$

$$\gamma_{yz} = \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = \frac{\partial u_{oz}}{\partial y} - \theta_2 \tag{14.1.3e}$$

$$\gamma_{zx} = \frac{\partial u_z}{\partial y} + \frac{\partial u_x}{\partial z} = \frac{\partial u_{oz}}{\partial x} - \theta_1 \tag{14.1.3f}$$

In Equations (14.13a–f), we have defined:

$$\varepsilon_{ox} = \frac{\partial u_{ox}}{\partial x} \tag{14.1.4a}$$

$$\varepsilon_{oy} = \frac{\partial u_{oy}}{\partial y} \tag{14.1.4b}$$

$$\gamma_{oxy} = \frac{\partial u_{ox}}{\partial y} + \frac{\partial u_{oy}}{\partial x} \tag{14.1.4c}$$

$$\varphi_{11} = \frac{\partial \theta_1}{\partial x} \tag{14.1.4d}$$

$$\varphi_{22} = \frac{\partial \theta_2}{\partial y} \tag{14.1.4e}$$

$$\varphi_{12} = \frac{1}{2}\left(\frac{\partial \theta_2}{\partial x} + \frac{\partial \theta_1}{\partial y}\right) \tag{14.1.4f}$$

The quantities defined by Equations (14.1.4a–f) are collectively termed the *generalized strains of a shell.*

Remark 14.1.1: Equations (14.1.2c) and (14.1.3d) imply that we *neglect* the effect of the strain ε_{zz}, although it is not exactly zero. We simply neglect its importance for the kinematics of the problem. The condition that we will actually enforce for a planar (flat) shell is that the stress σ_{zz} will be zero, where z is the axis perpendicular to the planar mid-surface of the shell. ■

Remark 14.1.2: The quantities ε_{ox}, ε_{oy}, and γ_{oxy} are termed the *membrane strain fields* of the shell, and they do quantify how much "in-plane stretching and distortion" an initially flat shell incurs due to the deformation. We can also define the symmetric, second-order *membrane strain tensor,* $[\varepsilon_0] = \begin{bmatrix} \varepsilon_{ox} & \varepsilon_{oxy} \\ \varepsilon_{oyx} & \varepsilon_{oy} \end{bmatrix}$, where $\varepsilon_{oij} = \frac{1}{2}\left(\frac{\partial u_{oi}}{\partial x_j} + \frac{\partial u_{oj}}{\partial x_i}\right)$,

$i, j = 1, 2$. This tensor follows the same exact transformation equations as the two-dimensional strain tensor in Section 7.7. ∎

Remark 14.1.3: The quantities φ_{11}, φ_{22}, and φ_{12} are termed the *curvature fields* of the shell, and they do quantify how much curved an initially flat shell becomes after the deformation. We can define a symmetric, second-order *curvature tensor*, $[\varphi] = \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix}$, where $\varphi_{ij} = \frac{1}{2}\left(\frac{\partial \theta_i}{\partial x_j} + \frac{\partial \theta_j}{\partial x_i}\right)$. This tensor also follows the same exact transformation equations as the two-dimensional strain tensor. ∎

14.2 Stress Resultants for Shells

For shells, it is convenient to express the effect of stresses in terms of *stress resultants*, which can be seen as a generalization of the internal loadings in a beam. Resultants are obtained after appropriate integration of stress terms over the thickness of the shell. Specifically, we can establish three components of *shell membrane forces*:

$$\hat{n}_{xx} = \int_{-d/2}^{d/2} \sigma_{xx} dz \tag{14.2.1a}$$

$$\hat{n}_{xy} = \int_{-d/2}^{d/2} \sigma_{xy} dz \tag{14.2.1b}$$

$$\hat{n}_{yy} = \int_{-d/2}^{d/2} \sigma_{yy} dz \tag{14.2.1c}$$

We can also define three components of *shell bending moments*:

$$\hat{m}_{xx} = -\int_{-d/2}^{d/2} \sigma_{xx} \cdot z dz \tag{14.2.2a}$$

$$\hat{m}_{yy} = -\int_{-d/2}^{d/2} \sigma_{yy} \cdot z dz \tag{14.2.2b}$$

$$\hat{m}_{xy} = -\int_{-d/2}^{d/2} \sigma_{xy} \cdot z dz \tag{14.2.2c}$$

Remark 14.2.1: The membrane forces have units of “force per unit width” of a shell section. Similarly, the shell moments have units of “moment per unit width” of a shell section. $\blacksquare$

Remark 14.2.2: We can define a symmetric, second-order *membrane force tensor*, $[\hat{n}] = \begin{bmatrix} \hat{n}_{xx} & \hat{n}_{xy} \\ \hat{n}_{yx} = \hat{n}_{xy} & \hat{n}_{yy} \end{bmatrix}$ and a symmetric, second-order *moment tensor*, $[\hat{m}] = \begin{bmatrix} \hat{m}_{xx} & \hat{m}_{xy} \\ \hat{m}_{yx} = \hat{m}_{xy} & \hat{m}_{yy} \end{bmatrix}$. $\blacksquare$

We can also establish the *shell shear forces per unit width*, $\hat{q}_x$ and $\hat{q}_y$, given by through-thickness integration of the contributions of the shear stresses σ_{xz} and σ_{yz}:

$$\hat{q}_x = \int_{-d/2}^{d/2} \sigma_{xy} dz = \sigma_{xz} \cdot d \qquad (14.2.3a)$$

$$\hat{q}_y = \int_{-d/2}^{d/2} \sigma_{yz} dz = \sigma_{yz} \cdot d \qquad (14.2.3b)$$

The representation of a segment of the mid-surface of the shell with the corresponding membrane force and bending moment/shear force stress resultants is shown in Figure 14.4. In accordance with the introductory descriptions of Section 14.1, the resistance of a shell to membrane forces is termed *membrane action*, while the bending resistance is also termed *plate action*.

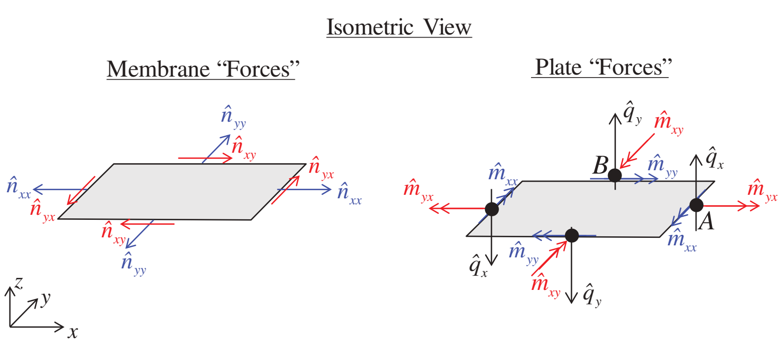


Figure 14.4 Stress resultants on a flat shell segment.

14.3 Differential Equations of Equilibrium and Boundary Conditions for Flat Shells

We will now obtain the strong form, that is, the differential equations and boundary conditions for a planar (flat) shell. *For planar shells, the governing equations for membrane action are independent of the corresponding expressions for plate action.* Let us first examine the membrane behavior of a shell. To this end, we isolate a small segment of a shell with planar dimensions (Δx) and (Δy), as shown in Figure 14.5, and examine its equilibrium. The figure includes membrane loads (forces per unit surface area) $p_x(x, y)$ and $p_y(x, y)$. Since the segment is small, the value of p_x and p_y is assumed to be constant.

We can write two equilibrium equations for the in-plane forces of the shell. One equation corresponds to equilibrium along the x-direction:

$$\sum F_x = 0 \rightarrow (n_{xx} + \Delta n_{xx}) \cdot \Delta y + \left(n_{xy} + \Delta n_{xy}\right) \cdot \Delta x - n_{xx} \cdot \Delta y - n_{xy} \cdot \Delta x + p_x \cdot \Delta x \cdot \Delta y = 0$$

$$\rightarrow \Delta n_{xx} \cdot \Delta y + \Delta n_{xy} \cdot \Delta x + p_x \cdot \Delta x \cdot \Delta y = 0 \rightarrow \frac{\Delta n_{xx}}{\Delta x} + \frac{\Delta n_{xy}}{\Delta y} + p_x = 0$$

If we take the limit as the size of the segment tends to zero, $\lim_{\substack{\Delta x \to 0 \\ \Delta x \to 0}}$, we obtain:

$$\frac{\partial n_{xx}}{\partial x} + \frac{\partial n_{xy}}{\partial y} + p_x = 0 \tag{14.3.1a}$$

We now write a second equilibrium equation, corresponding to forces along the y-direction:

$$\sum F_y = 0 \rightarrow \left(n_{yx} + \Delta n_{yx}\right) \cdot \Delta y + \left(n_{yy} + \Delta n_{yy}\right) \cdot \Delta x - n_{yx} \cdot \Delta y - n_{yy} \cdot \Delta x + p_y \cdot \Delta x \cdot \Delta y = 0$$

$$\rightarrow \Delta n_{yx} \cdot \Delta y + \Delta n_{yy} \cdot \Delta x + p_y \cdot \Delta x \cdot \Delta y = 0 \rightarrow \frac{\Delta n_{yx}}{\Delta x} + \frac{\Delta n_{yy}}{\Delta y} + p_y = 0$$

If we take the limit as the size of the piece tends to zero, $\lim_{\substack{\Delta x \to 0 \\ \Delta x \to 0}}$:

$$\frac{\partial n_{yx}}{\partial x} + \frac{\partial n_{yy}}{\partial y} + p_y = 0 \tag{14.3.1b}$$

Equations (14.3.1a,b) collectively comprise the system of partial differential equations of equilibrium for membrane (in-plane) loading of shells. These equations must be supplemented with essential and natural boundary conditions. The former correspond to prescribed values of u_{ox}, u_{oy} and are given by the expressions:

$$u_{ox} = \bar{u}_{ox} \text{ on } \Gamma_{ux} \tag{14.3.2a}$$

$$u_{oy} = \bar{u}_{oy} \text{ on } \Gamma_{uy} \tag{14.3.2b}$$

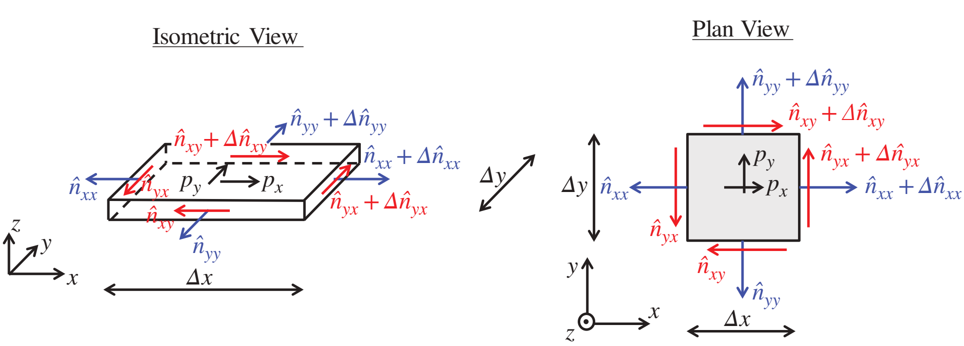


Figure 14.5 Membrane forces on shell segment.

The corresponding natural boundary conditions are written as follows.

$$\hat{n}_{xx} \cdot n_x + \hat{n}_{xy} \cdot n_y = \hat{f}_x \quad on \quad \Gamma_{tx} \tag{14.3.3a}$$

$$\hat{n}_{yx} \cdot n_x + \hat{n}_{yy} \cdot n_y = \hat{f}_y \quad on \quad \Gamma_{ty} \tag{14.3.3b}$$

where $\hat{f}_x$ and $\hat{f}_y$ are prescribed forces per unit length of the natural boundary and n_x, n_y are the components of the unit normal outward vector on the boundary of the shell.

Now let us continue with the differential equations and boundary conditions for plate (flexural) behavior of flat shells. To this end, we isolate a small segment of the shell with dimensions Δx, Δy, as shown in Figure 14.6, and examine the equilibrium for flexure. Figure 14.6 includes applied loads p_z (i.e., forces per unit surface area in the z-direction). We can establish three equilibrium equations, namely, force equilibrium along the z-axis and moment equilibrium about the x- and y- axes. Let us begin with the force equilibrium equation along the z-axis:

$$\sum F_z = 0 \rightarrow (\hat{q}_x + \Delta\hat{q}_x) \cdot \Delta y + \left(\hat{q}_y + \Delta\hat{q}_y\right) \cdot \Delta x - \hat{q}_x \cdot \Delta y - \hat{q}_y \cdot \Delta x + p_z \cdot \Delta x \cdot \Delta y = 0$$
$$\rightarrow \Delta\hat{q}_x \cdot \Delta y + \Delta\hat{q}_y \cdot \Delta x + p_z \cdot \Delta x \cdot \Delta y = 0 \rightarrow \frac{\Delta\hat{q}_x}{\Delta x} + \frac{\Delta\hat{q}_y}{\Delta y} + p_z = 0$$

If we take the limit as the size of the segment tends to zero, $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}$, we obtain:

$$\frac{\partial\hat{q}_x}{\partial x} + \frac{\partial\hat{q}_y}{\partial y} + p_z = 0 \tag{14.3.4a}$$

We continue with the equilibrium of moments about the y-axis, using point A in Figure 14.6 as the reference point:

$$\left(\sum M_y\right)^A = 0 \rightarrow (\hat{q}_x \cdot \Delta y) \cdot \Delta x + \hat{q}_y \cdot \frac{(\Delta x)^2}{2} - \left(\hat{q}_y + \Delta\hat{q}_y\right) \cdot \frac{(\Delta x)^2}{2} - p_z \cdot \Delta x \cdot \Delta y \cdot \frac{\Delta x}{2}$$
$$+ (\hat{m}_{xx} + \Delta\hat{m}_{xx}) \cdot \Delta y - \hat{m}_{xx} \cdot \Delta y + \left(\hat{m}_{xy} + \Delta\hat{m}_{xy}\right) \cdot \Delta x - \hat{m}_{xy} \cdot \Delta x = 0$$
$$\rightarrow \hat{q}_x \cdot \Delta x \cdot \Delta y + \Delta\hat{q}_y \frac{(\Delta x)^2}{2} - p_z \cdot \Delta x \cdot \Delta y \cdot \frac{\Delta x}{2} + \hat{m}_{xx} \cdot \Delta y + \Delta\hat{m}_{xy} \cdot \Delta x = 0$$
$$\rightarrow \hat{q}_x + \frac{\Delta\hat{q}_y \Delta x}{2\Delta y} - p_z \cdot \frac{\Delta x}{2} + \frac{\Delta\hat{m}_{xx}}{\Delta x} + \frac{\Delta\hat{m}_{xy}}{\Delta y} = 0$$

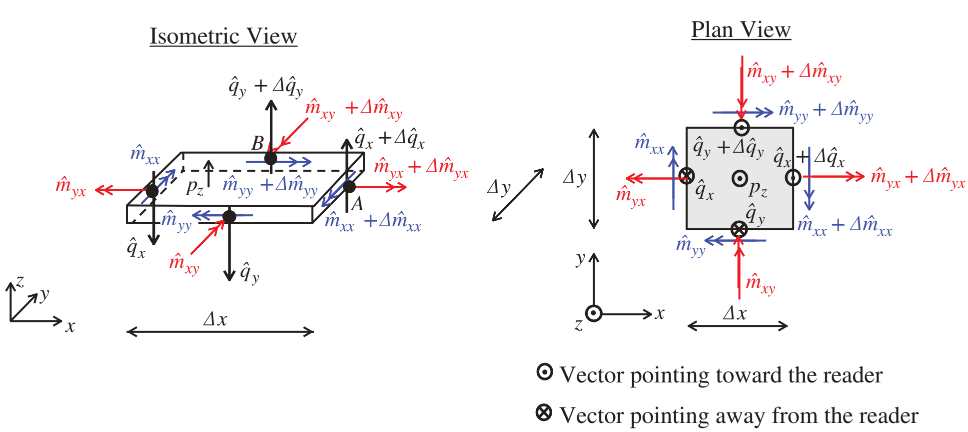


Figure 14.6 Plate bending forces on a shell segment.

If we take the limit as the size of the piece tends to zero, $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}$, we have:

$$\frac{\partial \hat{m}_{xx}}{\partial x} + \frac{\partial \hat{m}_{xy}}{\partial y} + \hat{q}_x = 0$$

where we have accounted for the fact that $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{\Delta \hat{q}_y \Delta x}{2\Delta y} \right) = \lim_{\Delta x \to 0} \left(\frac{\partial \hat{q}_y}{\partial y} \frac{\Delta x}{2} \right) = 0$, $\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(-p_z \cdot \frac{\Delta x}{2} \right) = 0$

In several cases, we also have an applied loading in the form of distributed moment per unit area, $\hat{m}_1$, which is applied in the direction shown in Figure 14.7. We thus can modify the obtained expression and write the general differential equation of moment equilibrium about the x-axis, for the case where we also have a distributed moment $\hat{m}_1$:

$$\frac{\partial \hat{m}_{xx}}{\partial x} + \frac{\partial \hat{m}_{xy}}{\partial y} + \hat{q}_x + \hat{m}_1 = 0 \qquad (14.3.4b)$$

We now write the equation of moment equilibrium about the x-axis, using point B in Figure 14.6 as the reference:

$$\left(\sum M_x\right)^B = 0 \rightarrow \left(\hat{q}_y \cdot \Delta x\right) \cdot \Delta y + \hat{q}_x \cdot \frac{(\Delta y)^2}{2} - (\hat{q}_x + \Delta \hat{q}_x) \cdot \frac{(\Delta y)^2}{2} - p_z \cdot \Delta x \cdot \Delta y \cdot \frac{\Delta y}{2}$$
$$+ \left(\hat{m}_{yy} + \Delta \hat{m}_{yy}\right) \cdot \Delta x - \hat{m}_{yy} \cdot \Delta x + \left(\hat{m}_{yx} + \Delta \hat{m}_{yx}\right) \cdot \Delta y - \hat{m}_{yy} \cdot \Delta y = 0$$
$$\rightarrow \hat{q}_y \cdot \Delta x \cdot \Delta y + \Delta \hat{q}_x \frac{(\Delta y)^2}{2} - p_z \cdot \Delta x \cdot \Delta y \cdot \frac{\Delta y}{2} + \hat{m}_{yy} \cdot \Delta x + \Delta \hat{m}_{yx} \cdot \Delta y = 0$$
$$\rightarrow \hat{q}_y + \frac{\Delta \hat{q}_x \Delta y}{2\Delta x_x} - p_z \cdot \frac{\Delta y}{2} + \frac{\Delta \hat{m}_{yy}}{\Delta y} + \frac{\Delta \hat{m}_{yx}}{\Delta x} = 0$$

If we take the limit as the size of the piece tends to zero, $\lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}}$,

we obtain: $\frac{\partial \hat{m}_{yx}}{\partial x} + \frac{\partial \hat{m}_{yy}}{\partial y} + \hat{q}_y = 0$

where we have accounted for the fact that

$\lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \left(\frac{\Delta \hat{q}_x \Delta y}{2\Delta x} \right) = \lim_{\Delta y \to 0} \left(\frac{\partial \hat{q}_x}{\partial x} \frac{\Delta y}{2} \right) = 0$, $\lim_{\substack{\Delta y \to 0 \\ \Delta x \to 0}} \left(-p_z \cdot \frac{\Delta y}{2} \right) = 0.$

We will once again write the differential equation for the most general case, where we also have a distributed moment per unit area, $\hat{m}_2$, as shown in Figure 14.7:

$$\frac{\partial \hat{m}_{yx}}{\partial x} + \frac{\partial \hat{m}_{yy}}{\partial y} + \hat{q}_y + \hat{m}_2 = 0 \qquad (14.3.4c)$$

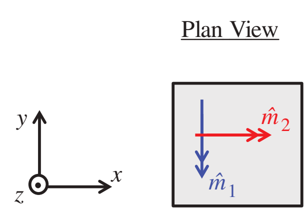


Figure 14.7 Applied distributed moments (per unit area) in a shell.

In summary, the flexural (bending or plate) response of a flat shell is described by the system of differential equations (14.3.4a–c).

The essential boundary conditions for flexural (plate) action can be written as follows.

$$u_{oz} = \bar{u}_{oz} \quad on \quad \Gamma_{uz} \tag{14.3.5a}$$

$$\theta_1 = \bar{\theta}_1 \quad on \quad \Gamma_{u\theta 1} \tag{14.3.5b}$$

$$\theta_2 = \bar{\theta}_2 \quad on \quad \Gamma_{u\theta 2} \tag{14.3.5c}$$

The natural boundary conditions for plate behavior of a planar shell are:

$$\hat{q}_x \cdot n_x + \hat{q}_y \cdot n_y = \bar{\hat{q}} \; on \; \Gamma_{tz} \tag{14.3.6a}$$

$$\hat{m}_{xx} \cdot n_x + \hat{m}_{xy} \cdot n_y = \bar{\hat{m}}_x \quad on \quad \Gamma_{t\theta 1} \tag{14.3.6b}$$

$$\hat{m}_{yx} \cdot n_x + \hat{m}_{yy} \cdot n_y = \bar{\hat{m}}_y \quad on \quad \Gamma_{t\theta 2} \tag{14.3.6c}$$

Remark 14.3.1: The differential equations and boundary conditions that we obtained in this Section are not based on an assumption regarding the material behavior, thus they are valid for all types of material! ■

Remark 14.3.2: A special note is necessary for the natural boundary conditions. In real life, we most frequently have prescribed tractions normal and tangential to a boundary segment, which leads to corresponding directions of the membrane forces and moments in the natural boundary. ■

14.4 Constitutive Law for Linear Elasticity in Terms of Stress Resultants and Generalized Strains

As mentioned in Section 14.1, a basic assumption made for the theory of flat (planar) shells is that $\sigma_{zz} = 0$. This condition is referred to as the *quasi-plane-stress assumption* and allows us to establish a reduced constitutive (stress-strain) law for linearly elastic materials in shells. Let us revisit the generic stress-strain law for linear elasticity, provided in Section 7.5.

$$\{\sigma\} = [D]\{\varepsilon\} \tag{14.4.1}$$

where $\{\sigma\} = \begin{bmatrix}\sigma_{xx} & \sigma_{yy} & \sigma_{zz} & \sigma_{xy} & \sigma_{yz} & \sigma_{zx}\end{bmatrix}^T$, $\{\varepsilon\} = \begin{bmatrix}\varepsilon_{xx} & \varepsilon_{yy} & \varepsilon_{zz} & 2\varepsilon_{xy} & 2\varepsilon_{yz} & 2\varepsilon_{zx}\end{bmatrix}^T$ and $[D]$ is the (6×6) material stiffness matrix.

To account for the quasi-plane stress assumption, we will first expand the third row of Equation (14.4.1), which yields the *zz*-stress component:

$$\sigma_{zz} = D_{31}\varepsilon_{xx} + D_{32}\varepsilon_{yy} + D_{33}\varepsilon_{zz} + D_{34} \cdot 2\varepsilon_{xy} + D_{35} \cdot 2\varepsilon_{yz} + D_{36} \cdot 2\varepsilon_{zx} \tag{14.4.2}$$

Since we know that the left-hand side of Equation (14.4.2) is zero, $\sigma_{zz} = 0$, we can rearrange the equation to express the strain component ε_{zz} in terms of the other strain components:

$$D_{31}\varepsilon_{xx} + D_{32}\varepsilon_{yy} + D_{33}\varepsilon_{zz} + D_{34} \cdot 2\varepsilon_{xy} + D_{35} \cdot 2\varepsilon_{yz} + D_{36} \cdot 2\varepsilon_{zx} = 0$$

$$\rightarrow \varepsilon_{zz} = -\frac{1}{D_{33}}\left(D_{31}\varepsilon_{xx} + D_{32}\varepsilon_{yy} + D_{34} \cdot 2\varepsilon_{xy} + D_{35} \cdot 2\varepsilon_{yz} + D_{36} \cdot 2\varepsilon_{zx}\right) \tag{14.4.3}$$

We will now define a *reduced stress vector,* $\{\tilde{\sigma}\}$:

$$\{\tilde{\sigma}\} = \left[\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{zz} \;\; \sigma_{xy} \;\; \sigma_{yz} \;\; \sigma_{zx}\right]^T \tag{14.4.4a}$$

which includes the stress components except for $\sigma_{zz} = 0$. We also define the corresponding *reduced strain vector,*

$$\{\tilde{\varepsilon}\} = \left[\varepsilon_{xx} \;\; \varepsilon_{yy} \;\; 2\varepsilon_{xy} \;\; 2\varepsilon_{yz} \;\; 2\varepsilon_{zx}\right]^T \tag{14.4.4b}$$

We will establish the constitutive law for a linearly elastic material in a shell, in terms of the reduced vectors $\{\tilde{\sigma}\}$ and $\{\tilde{\varepsilon}\}$.

First of all, we note that we can isolate the rows of the matrix Equation (14.4.1) corresponding to the components of $\{\tilde{\sigma}\}$:

$$\{\sigma\} = \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = [D_{red}]\{\varepsilon\} \tag{14.4.5}$$

where $[D_{red}]$ is a *reduced* version of the material stiffness matrix $[D]$, obtained after removing the third row of that matrix (the specific row corresponds to Equation 14.4.2):

$$[D_{red}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{21} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{41} & D_{42} & D_{43} & D_{44} & D_{45} & D_{46} \\ D_{51} & D_{52} & D_{53} & D_{54} & D_{55} & D_{56} \\ D_{61} & D_{62} & D_{63} & D_{64} & D_{65} & D_{66} \end{bmatrix} \tag{14.4.6}$$

Given Equation (14.4.3), we can write a *matrix transformation relation between* $\{\varepsilon\}$ *and* $\{\tilde{\varepsilon}\}$:

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{D_{31}}{D_{33}} & -\frac{D_{32}}{D_{33}} & -\frac{D_{34}}{D_{33}} & -\frac{D_{35}}{D_{33}} & -\frac{D_{36}}{D_{33}} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ 2\varepsilon_{xy} \\ 2\varepsilon_{yz} \\ 2\varepsilon_{zx} \end{Bmatrix} = \left[T_{pl-s}\right]\{\tilde{\varepsilon}\} \tag{14.4.7}$$

where

$$\left[T_{pl-s}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\dfrac{D_{31}}{D_{33}} & -\dfrac{D_{32}}{D_{33}} & -\dfrac{D_{34}}{D_{33}} & -\dfrac{D_{35}}{D_{33}} & -\dfrac{D_{36}}{D_{33}} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{14.4.8}$$

is a (6 × 5) transformation matrix giving the full strain vector $\{\varepsilon\}$ in terms of the reduced strain vector $\{\tilde{\varepsilon}\}$.

We can now plug Equation (14.4.7) into Equation (14.4.5) to finally obtain:

$$\{\tilde{\sigma}\} = [D_{red}]\left[T_{pl-s}\right]\{\tilde{\varepsilon}\} \rightarrow \{\tilde{\sigma}\} = \left[\tilde{D}\right]\{\tilde{\varepsilon}\} \tag{14.4.9}$$

where $\left[\tilde{D}\right] = [D_{red}]\left[T_{pl-s}\right]$ is a reduced (*condensed*) material stiffness matrix, expressing the relation between the stress components of $\{\tilde{\sigma}\}$ (i.e., the stress components that can attain nonzero values) and the corresponding strain components of $\{\tilde{\varepsilon}\}$. We can directly write an equation providing each of the components of $\left[\tilde{D}\right]$ in terms of the corresponding components of $[D]$:

$$\tilde{D}_{ij} = D_{ij} - D_{i3}\frac{D_{3j}}{D_{33}} \tag{14.4.10}$$

The remainder of this section will consider the simplest possible case of an isotropic, linearly elastic material, in which the reduced constitutive law only includes two elastic constants, namely, the modulus of elasticity E and Poisson's ratio ν. In this case, $[D]$ is given by Equation (7.5.10b), and one can verify that Equation (14.4.9) attains the form:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \tag{14.4.11}$$

Now, since the stress resultants can be obtained by through-thickness integration of appropriate stress components, and since the latter can be obtained from the strain components (through Equation 14.4.11), we can establish a relation between the stress resultants and the generalized strains defined in Equations (14.1.4a–f). In our derivation, we will assume that the elastic constants (E and ν) can vary over the shell surface, that is, they are functions of x and y, but they do not change over the thickness at a given location of the shell; thus, E and ν are independent of z!

We will first work with the bending moment $\hat{m}_{xx}$. From Equation (14.2.2a), we had:

$$\hat{m}_{xx} = -\int_{-d/2}^{d/2} \sigma_{xx} \cdot z dz.$$

We use Equation (14.4.11), to express the stress σ_{xx} in terms of the strains:

$$\sigma_{xx} = \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{E\cdot\nu}{1-\nu^2}\varepsilon_{yy} \tag{14.4.12}$$

We can then plug Equations (14.1.3a–b), giving the strains ε_{xx} and ε_{yy} in terms of the generalized strains, into Equation (14.4.12):

$$\sigma_{xx} = \frac{E}{1-\nu^2}(\varepsilon_{ox} - \varphi_{11}\cdot z) + \frac{E\cdot\nu}{1-\nu^2}(\varepsilon_{oy} - \varphi_{22}\cdot z) \tag{14.4.13}$$

If we plug Equation (14.4.13) into Equation (14.2.2a), we eventually obtain:

$$\hat{m}_{xx} = -\frac{E}{1-\nu^2}\int_{-d/2}^{d/2}\varepsilon_{ox}\cdot zdz - \frac{E}{1-\nu^2}\int_{-d/2}^{d/2}(-\varphi_{11}\cdot z)\cdot zdz - \frac{E\cdot\nu}{1-\nu^2}\int_{-d/2}^{d/2}\varepsilon_{oy}\cdot zdz - \frac{E\cdot\nu}{1-\nu^2}\int_{-d/2}^{d/2}(-\varphi_{22}\cdot z)\cdot zdz$$

$$= -\frac{E}{1-\nu^2}\cdot\varepsilon_{ox}\int_{-d/2}^{d/2} zdz + \frac{E}{1-\nu^2}\cdot\varphi_{11}\int_{-d/2}^{d/2} z^2dz - \frac{E\cdot\nu}{1-\nu^2}\cdot\varepsilon_{oy}\int_{-d/2}^{d/2} zdz + \frac{E\cdot\nu}{1-\nu^2}\cdot\varphi_{22}\int_{-d/2}^{d/2} z^2dz$$

Finally, if we account for Equation (14.1.1), that is, $\int_{-d/2}^{d/2} zdz = 0$, we have:

$$\hat{m}_{xx} = \frac{E}{1-\nu^2}\cdot\frac{d^3}{12}\varphi_{11} + \frac{E\cdot\nu}{1-\nu^2}\cdot\frac{d^3}{12}\varphi_{22} \tag{14.4.14}$$

Along the same lines, the moment $\hat{m}_{yy}$ is obtained from Equation (14.2.2b), $\hat{m}_{yy} = -\int_{-d/2}^{d/2}\sigma_{yy}\cdot zdz$, where the stress σ_{yy} can be expressed in terms of the generalized strains, by means of Equations (14.4.11) and (14.1.3a–b):

$$\sigma_{yy} = \frac{E\cdot\nu}{1-\nu^2}\varepsilon_{xx} + \frac{E}{1-\nu^2}\varepsilon_{yy} = \frac{E\cdot\nu}{1-\nu^2}(\varepsilon_{ox} - \varphi_{11}\cdot z) + \frac{E}{1-\nu^2}(\varepsilon_{oy} - \varphi_{22}\cdot z) \tag{14.4.15}$$

Plugging Equation (14.4.15) into Equation (14.2.2b) yields:

$$\hat{m}_{yy} = -\frac{E\cdot\nu}{1-\nu^2}\int_{-d/2}^{d/2}\varepsilon_{ox}\cdot zdz - \frac{E\cdot\nu}{1-\nu^2}\int_{-d/2}^{d/2}(-\varphi_{11}\cdot z)\cdot zdz$$

$$-\frac{E}{1-\nu^2}\int_{-d/2}^{d/2}\varepsilon_{oy}\cdot zdz - \frac{E}{1-\nu^2}\int_{-d/2}^{d/2}(-\varphi_{22}\cdot z)\cdot zdz$$

$$= -\frac{E\cdot\nu}{1-\nu^2}\cdot\varepsilon_{ox}\int_{-d/2}^{d/2} zdz + \frac{E\cdot\nu}{1-\nu^2}\cdot\varphi_{11}\int_{-d/2}^{d/2} z^2dz - \frac{E}{1-\nu^2}\cdot\varepsilon_{oy}\int_{-d/2}^{d/2} zdz + \frac{E}{1-\nu^2}\cdot\varphi_{22}\int_{-d/2}^{d/2} z^2dz$$

and if we account for Equation (14.1.1), we have:

$$\hat{m}_{yy} = \frac{E \cdot \nu}{1-\nu^2} \cdot \frac{d^3}{12}\varphi_{11} + \frac{E}{1-\nu^2} \cdot \frac{d^3}{12}\varphi_{22} \tag{14.4.16}$$

We continue with the expression for the moment $\hat{m}_{xy}$; we use Equations (14.2.2c), (14.4.11), and (14.1.3c), and also set $G = \frac{E}{2(1+\nu)}$, to obtain:

$$\hat{m}_{xy} = -\int_{-d/2}^{d/2} \sigma_{xy} \cdot z dz = -\int_{-d/2}^{d/2} G \cdot \gamma_{xy} \cdot z dz = -\int_{-d/2}^{d/2} G\left(\gamma_{oxy} - 2\varphi_{12} \cdot z\right) \cdot z dz$$

$$\hat{m}_{xy} = -G\int_{-d/2}^{d/2} \gamma_{oxy} \cdot z dz - G\int_{-d/2}^{d/2} (-2\varphi_{12} \cdot z) \cdot z dz = -G \cdot \gamma_{oxy}\int_{-d/2}^{d/2} z dz + G \cdot 2\varphi_{12}\int_{-d/2}^{d/2} z^2 dz$$

Accounting for Equation (14.1.1) yields:

$$\hat{m}_{xy} = 2G \cdot \frac{d^3}{12}\varphi_{12} \tag{14.4.17}$$

For the through-thickness shear force $\hat{q}_x$ we use Equations (14.2.3a), (14.4.11), and (14.1.3f):

$$\hat{q}_x = \int_{-d/2}^{d/2} \sigma_{zx} dz = \int_{-d/2}^{d/2} G \cdot \gamma_{zx} dz$$

We will finally write:

$$\hat{q}_x = \kappa \cdot G \cdot d \cdot \gamma_{zx} \tag{14.4.18}$$

where, just like for Timoshenko beam theory in Section 13.8, we have used a shear correction factor, κ, (equal to 5/6 for linear elasticity) to improve the accuracy of the results using the Reissner-Mindlin theory. The shear correction factor is necessary, because the simplifying assumption that the through-thickness shear stresses are constant over the thickness is not perfectly accurate.

Finally, for the shear force $\hat{q}_y$, we use Equations (14.2.3b), (14.4.11), and (14.1.3e):

$$\hat{q}_y = \int_{-d/2}^{d/2} \sigma_{yz} dz = \int_{-d/2}^{d/2} G \cdot \gamma_{yz} dz = G\int_{-d/2}^{d/2} \gamma_{yz} dz \rightarrow \hat{q}_y = \kappa \cdot G \cdot d \cdot \gamma_{yz} \tag{14.4.19}$$

Equations (14.4.14), (14.4.16–19), collectively define a *generalized stress-strain relation for bending (plate) behavior*:

$$\{\hat{\sigma}_b\} = \left[\hat{D}_b\right]\{\hat{\varepsilon}_b\} \tag{14.4.20}$$

where $\{\hat{\sigma}_b\}$ is the *generalized stress vector for bending (plate) behavior*:

$$\{\hat{\sigma}_b\} = \left[\hat{m}_{xx} \quad \hat{m}_{yy} \quad \hat{m}_{xy} \quad \hat{q}_x \quad \hat{q}_y\right]^T \tag{14.4.21}$$

$\{\hat{\varepsilon}_b\}$ is the corresponding *generalized strain vector for bending behavior*:

$$\{\hat{\varepsilon}_b\} = \left[\varphi_{11} \;\; \varphi_{22} \;\; 2\varphi_{12} \;\; \gamma_{zx} \;\; \gamma_{zy}\right]^T \tag{14.4.22}$$

and $\left[\hat{D}_b\right]$ is the *generalized stiffness matrix for bending behavior and for linearly elastic isotropic material*:

$$\left[\hat{D}_b\right] = \begin{bmatrix} \frac{E\cdot d^3}{12(1-\nu^2)} & \frac{\nu\cdot E\cdot d^3}{12(1-\nu^2)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & G\cdot\frac{d^3}{12} & 0 & 0 \\ 0 & 0 & 0 & \kappa\cdot G\cdot d & 0 \\ 0 & 0 & 0 & 0 & \kappa\cdot G\cdot d \end{bmatrix} \tag{14.4.23}$$

It is also possible to write Equation (14.4.20) in a *partitioned form*, separating the moment (flexural) and shear-force terms:

$$\begin{Bmatrix} \{\hat{\sigma}_f\} \\ \{\hat{\sigma}_s\} \end{Bmatrix} = \begin{Bmatrix} \left[\hat{D}_f\right]\{\varepsilon_f\} \\ \left[\hat{D}_s\right]\{\varepsilon_s\} \end{Bmatrix} = \begin{bmatrix} \left[\hat{D}_f\right] & [0] \\ [0] & \left[\hat{D}_s\right] \end{bmatrix} \begin{Bmatrix} \{\hat{\varepsilon}_f\} \\ \{\hat{\varepsilon}_s\} \end{Bmatrix} \tag{14.4.24}$$

where

$$\{\hat{\sigma}_f\} = \left[\hat{m}_{xx} \;\; \hat{m}_{yy} \;\; \hat{m}_{xy}\right]^T \tag{14.4.25}$$

$$\{\hat{\sigma}_s\} = \left[\hat{q}_x \;\; \hat{q}_y\right]^T \tag{14.4.26}$$

$$\{\hat{\varepsilon}_f\} = \left[\varphi_{11} \;\; \varphi_{22} \;\; 2\varphi_{12}\right]^T \tag{14.4.27}$$

$$\{\hat{\varepsilon}_s\} = \left[\gamma_{zx} \;\; \gamma_{zy}\right]^T \tag{14.4.28}$$

$$\left[\hat{D}_f\right] = \begin{bmatrix} \frac{E\cdot d^3}{12(1-\nu^2)} & \frac{\nu\cdot E\cdot d^3}{12(1-\nu^2)} & 0 \\ \frac{\nu\cdot E\cdot d^3}{12(1-\nu^2)} & \frac{E\cdot d^3}{12(1-\nu^2)} & 0 \\ 0 & 0 & G\cdot\frac{d^3}{12} \end{bmatrix} = \frac{E\cdot d^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{14.4.29}$$

and

$$\left[\hat{D}_s\right] = \begin{bmatrix} \kappa\cdot G\cdot d & 0 \\ 0 & \kappa\cdot G\cdot d \end{bmatrix} = \kappa\cdot G\cdot d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{14.4.30}$$

We will now establish a generalized stress-strain law for membrane (in-plane) behavior and linearly elastic, isotropic material. We begin with the membrane force component $\hat{n}_{xx}$. We use Equations (14.2.1a), (14.1.4a–b), and (14.4.11), to obtain:

$$\hat{n}_{xx} = \int_{-d/2}^{d/2} \sigma_{xx} dz = \int_{-d/2}^{d/2} \left[\frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{E\cdot\nu}{1-\nu^2}\varepsilon_{yy}\right] dz$$

$$= \int_{-d/2}^{d/2} \left[\frac{E}{1-\nu^2}(\varepsilon_{ox} - \varphi_{11}\cdot z) + \frac{E\cdot\nu}{1-\nu^2}(\varepsilon_{oy} - \varphi_{22}\cdot z)\right] dz$$

$$= \frac{E}{1-\nu^2}\int_{-d/2}^{d/2} \varepsilon_{ox} dz + \frac{E}{1-\nu^2}\int_{-d/2}^{d/2} (-\varphi_{11})\cdot zdz + \frac{E\cdot\nu}{1-\nu^2}\int_{-d/2}^{d/2} \varepsilon_{oy} dz + \frac{E\cdot\nu}{1-\nu^2}\int_{-d/2}^{d/2} (-\varphi_{22})\cdot zdz$$

$$= \frac{E}{1-\nu^2}\cdot\varepsilon_{ox}\int_{-d/2}^{d/2} dz - \frac{E}{1-\nu^2}\cdot\varphi_{11}\int_{-d/2}^{d/2} zdz + \frac{E\cdot\nu}{1-\nu^2}\cdot\varepsilon_{oy}\int_{-d/2}^{d/2} dz - \frac{E\cdot\nu}{1-\nu^2}\cdot\varphi_{22}\int_{-d/2}^{d/2} zdz$$

If we account for Equation (14.1.1), we have:

$$\hat{n}_{xx} = \frac{E}{1-\nu^2}\cdot d\cdot\varepsilon_{ox} + \frac{E\cdot\nu}{1-\nu^2}\cdot d\cdot\varepsilon_{oy} \tag{14.4.31}$$

If we use Equations (14.2.1b), (14.1.4a–b), and (14.4.11), and also account for Equation (14.1.1), we obtain a similar equation for the membrane force $\hat{n}_{yy}$:

$$\hat{n}_{yy} = \frac{E\cdot\nu}{1-\nu^2}\cdot d\cdot\varepsilon_{ox} + \frac{E}{1-\nu^2}\cdot d\cdot\varepsilon_{oy} \tag{14.4.32}$$

Finally, we use Equations (14.2.1c), (14.1.4c), and (14.4.11), to establish an expression for the force $\hat{n}_{xy}$:

$$\hat{n}_{xy} = \int_{-d/2}^{d/2} \sigma_{xy} dz = \int_{-d/2}^{d/2} G\cdot\gamma_{xy} dy = \int_{-d/2}^{d/2} G\left(\gamma_{oxy} - 2\varphi_{12}\cdot z\right) dz$$

$$= G\int_{-d/2}^{d/2} \gamma_{oxy} dz + G\int_{-d/2}^{d/2} (-2\varphi_{12})\cdot zdz = G\cdot\gamma_{oxy}\int_{-d/2}^{d/2} dz - G\cdot 2\varphi_{12}\int_{-d/2}^{d/2} zdz$$

and, if we account for Equation (14.1.1):

$$\hat{n}_{xy} = G\cdot d\cdot\gamma_{oxy} \tag{14.4.33}$$

We can now combine Equations (14.4.31), (14.4.32), and (14.4.33) into a unique matrix expression:

$$\{\hat{\sigma}_m\} = \left[\hat{D}_m\right]\{\hat{\varepsilon}_m\} \tag{14.4.34}$$

where

$$\{\hat{\varepsilon}_m\} = \begin{bmatrix}\varepsilon_{ox} & \varepsilon_{oy} & \varepsilon_{oxy}\end{bmatrix}^T \tag{14.4.35}$$

is the generalized membrane strain vector,

$$\{\hat{\sigma}_m\} = \begin{bmatrix}\hat{n}_{xx} & \hat{n}_{yy} & \hat{n}_{xy}\end{bmatrix}^T \tag{14.4.36}$$

is the generalized stress vector, and

$$\left[\hat{D}_m\right] = \frac{E \cdot d}{1-\nu^2}\begin{bmatrix}1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix} \tag{14.4.37}$$

Remark 14.4.1: The constitutive law for membrane behavior is the same as the one we had obtained for two-dimensional plane-stress elasticity, except that the stiffness terms are multiplied by the thickness d! ■

Remark 14.4.2: It is important to keep in mind that, if the material elastic constants vary over the thickness of the shell, that is, E and ν are functions of z, then there is *coupling* between the bending and membrane behavior. For example, the generalized membrane strains will affect the value of the generalized bending stresses, and vice versa. In that case, we generally cannot write separate constitutive laws for the membrane stress resultants and for the bending stress resultants. ■

Now, we will briefly discuss the most general situation, where we have linearly elastic (but not necessarily isotropic) material. In this case, we can always write a constitutive relation of the form:

$$\{\hat{\sigma}\} = \left[\hat{D}\right]\{\hat{\varepsilon}\} \tag{14.4.38}$$

where $\{\hat{\varepsilon}\}$ is the generalized strain vector of the shell (containing all the generalized strains):

$$\{\hat{\varepsilon}\} = \begin{bmatrix}\varepsilon_{ox} & \varepsilon_{oy} & \gamma_{oxy} & \varphi_{11} & \varphi_{22} & 2\varphi_{12} & \gamma_{zx} & \gamma_{zy}\end{bmatrix}^T \tag{14.4.39}$$

and $\{\hat{\sigma}\}$ is the generalized stress vector of the shell (containing all the generalized stresses):

$$\{\hat{\sigma}\} = \begin{bmatrix}\hat{n}_{xx} & \hat{n}_{yy} & \hat{n}_{xy} & \hat{m}_{xx} & \hat{m}_{yy} & \hat{m}_{xy} & \hat{q}_x & \hat{q}_y\end{bmatrix}^T \tag{14.4.40}$$

The special case of isotropic linear elasticity (and for material constants E and ν, which do not vary with z) leads to generalized stress-strain relations for membrane terms, flexural terms and shear terms that are uncoupled from each other; this in turn allows us to cast Equation (14.4.38) in the following diagonal, block-matrix form:

$$\{\hat{\sigma}\} = \begin{Bmatrix} \{\hat{\sigma}_m\} \\ \{\hat{\sigma}_f\} \\ \{\hat{\sigma}_s\} \end{Bmatrix} = \begin{bmatrix} [\hat{D}_m] & [0] & [0] \\ [0] & [\hat{D}_f] & [0] \\ [0] & [0] & [\hat{D}_s] \end{bmatrix} \begin{Bmatrix} \{\hat{\varepsilon}_m\} \\ \{\hat{\varepsilon}_f\} \\ \{\hat{\varepsilon}_s\} \end{Bmatrix} \tag{14.4.41}$$

Thus, the case of isotropic linear elasticity can be considered as a special case of Equation (14.4.38), with:

$$[\hat{D}] = \begin{bmatrix} [\hat{D}_m] & [0] & [0] \\ [0] & [\hat{D}_f] & [0] \\ [0] & [0] & [\hat{D}_s] \end{bmatrix} \tag{14.4.42}$$

14.5 Weak Form of Shell Equations

We will now proceed to obtain the weak form for flat shells, using the same conceptual procedure as that employed for elasticity and beam problems. We first establish an arbitrary vector field $\{w\} = \begin{bmatrix} w_{ox} & w_{oy} & w_{oz} & w_{\theta 1} & w_{\theta 2} \end{bmatrix}^T$, with each of the components vanishing at the corresponding essential boundary segment:

$$w_{ox} = 0 \text{ at } \Gamma_{ux} \tag{14.5.1a}$$

$$w_{oy} = 0 \text{ at } \Gamma_{uy} \tag{14.5.1b}$$

$$w_{oz} = 0 \text{ at } \Gamma_{uz} \tag{14.5.1c}$$

$$w_{\theta 1} = 0 \text{ at } \Gamma_{u\theta 1} \tag{14.5.1d}$$

$$w_{\theta 2} = 0 \text{ at } \Gamma_{u\theta 2} \tag{14.5.1e}$$

We then multiply each of the five differential equations of equilibrium—that is, Equations (14.3.1a,b) for membrane action and Equations (14.3.4a,b,c) for plate behavior—by the corresponding component of the $\{w\}$-vector. Subsequently, we integrate each expression over the two-dimensional domain defined by the mid-surface of the shell structure and obtain the weak form. There is a more straightforward way to obtain the weak form, namely, the *continuum-based formulation*, in which we use the (known) weak form for three-dimensional elasticity (which was presented in Section 9.1 and must apply for any three-dimensional solid elastic body), then impose the kinematic constraints that we have for shell theory, as well as the condition $\sigma_{zz} = 0$. Since $\{w\}$ can be thought of as an arbitrary, virtual vector field with mid-surface deformations, we can define the corresponding three-dimensional virtual displacement components, such that they satisfy the kinematic assumptions of shell theory:

$$w_x = (x,y,z) = w_{ox}(x,y) - z \cdot w_{\theta 1} \tag{14.5.2a}$$

$$w_y = (x,y,z) = w_{oy}(x,y) - z \cdot w_{\theta 2} \tag{14.5.2b}$$

$$w_z(x,y,z) \approx w_{oz}(x,y) \tag{14.5.2c}$$

We can also establish the virtual strain components, obtained from partial differentiation of the virtual displacement field:

$$\bar{\varepsilon}_{xx} = \frac{\partial w_x}{\partial x} = \frac{\partial w_{ox}}{\partial x} - z \cdot \frac{\partial w_{\theta 1}}{\partial x} = \bar{\varepsilon}_{ox} - z \cdot \bar{\varphi}_{11} \quad (14.5.3a)$$

$$\bar{\varepsilon}_{yy} = \frac{\partial w_y}{\partial y} = \frac{\partial w_{oy}}{\partial y} - z \cdot \frac{\partial w_{\theta 2}}{\partial y} = \bar{\varepsilon}_{oy} - z \cdot \bar{\varphi}_{22} \quad (14.5.3b)$$

$$\bar{\gamma}_{xy} = \frac{\partial w_x}{\partial y} + \frac{\partial w_y}{\partial x} = \frac{\partial w_{ox}}{\partial y} + \frac{\partial w_{oy}}{\partial x} - z \cdot \frac{\partial w_{\theta 1}}{\partial y} - z \cdot \frac{\partial w_{\theta 2}}{\partial x} = \bar{\gamma}_{oxy} - z \cdot 2\bar{\varphi}_{12} \quad (14.5.3c)$$

$$\bar{\varepsilon}_{zz} = \frac{\partial w_z}{\partial z} \approx 0 \quad (14.5.3d)$$

$$\bar{\gamma}_{yz} = \frac{\partial w_z}{\partial y} + \frac{\partial w_y}{\partial z} = \frac{\partial w_{oz}}{\partial y} - w_{\theta 2} \quad (14.5.3e)$$

$$\bar{\gamma}_{zx} = \frac{\partial w_z}{\partial x} + \frac{\partial w_x}{\partial z} = \frac{\partial w_{oz}}{\partial x} - w_{\theta 1} \quad (14.5.3f)$$

In Equations (14.5.3a–f), we have used the following *virtual generalized strain components:*

$$\bar{\varepsilon}_{ox} = \frac{\partial w_{ox}}{\partial x} \quad (14.5.4a)$$

$$\bar{\varepsilon}_{oy} = \frac{\partial w_{oy}}{\partial y} \quad (14.5.4b)$$

$$\bar{\gamma}_{oxy} = \frac{\partial w_{ox}}{\partial y} + \frac{\partial w_{oy}}{\partial x} \quad (14.5.4c)$$

$$\bar{\varphi}_{11} = \frac{\partial w_{\theta 1}}{\partial x} \quad (14.5.4d)$$

$$\bar{\varphi}_{22} = \frac{\partial w_{\theta 2}}{\partial y} \quad (14.5.4e)$$

$$\bar{\varphi}_{12} = \frac{1}{2}\left(\frac{\partial w_{\theta 2}}{\partial x} + \frac{\partial w_{\theta 1}}{\partial y}\right) \quad (14.5.4f)$$

We will now examine the *internal virtual work* $\bar{W}_{int}$, that is, the left-hand side in the weak form for three-dimensional elasticity in Box 9.1.1. If the three-dimensional domain defined by the volume of the shell is denoted by V and the domain defined by the shell mid-surface is denoted by Ω, as shown in Figure 14.8, then we can conduct the volume integration into two stages, one stage corresponding to integration over the thickness (integration with z) and the other to integration over Ω, then the internal virtual work is given by:

$$\bar{W}_{int} = \iiint_V \left(\bar{\varepsilon}_{xx} \cdot \sigma_{xx} + \bar{\varepsilon}_{yy} \cdot \sigma_{yy} + \bar{\gamma}_{xy} \cdot \sigma_{xy} + \bar{\gamma}_{yz} \cdot \sigma_{yz} + \bar{\gamma}_{zx} \cdot \sigma_{zx}\right) dV \rightarrow$$

$$\rightarrow = \iint_\Omega \left[\int_{-d/2}^{d/2} \left(\bar{\varepsilon}_{xx} \cdot \sigma_{xx} + \bar{\varepsilon}_{yy} \cdot \sigma_{yy} + \bar{\gamma}_{xy} \cdot \sigma_{xy} + \bar{\gamma}_{yz} \cdot \sigma_{yz} + \bar{\gamma}_{zx} \cdot \sigma_{zx}\right) dz\right] d\Omega \quad (14.5.5)$$

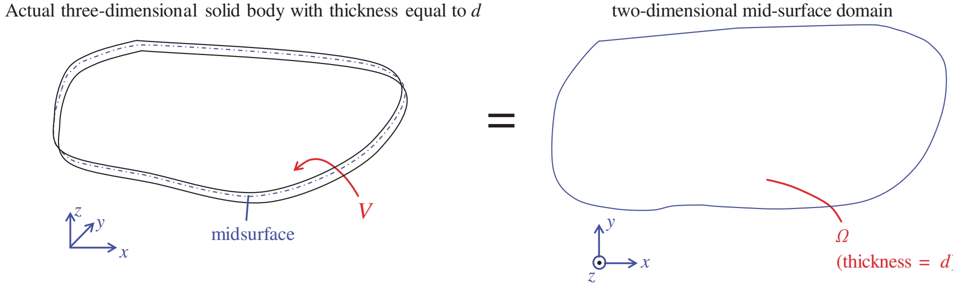


Figure 14.8 Conversion of a three-dimensional volume integral to a surface integral of a shell mid-surface.

Now, let us examine each one of the five internal virtual work terms in the integrand of the left-hand side of Equation (14.5.5) and see how it can be expressed using the loading and deformation quantities of the mid-surface. The first term in Equation (14.5.5) is:

$$\int_{-d/2}^{d/2} \bar{\varepsilon}_{xx}\cdot\sigma_{xx}dz = \int_{-d/2}^{d/2}(\bar{\varepsilon}_{ox}-\bar{\varphi}_{11}\cdot z)\cdot\sigma_{xx}dz = \int_{-d/2}^{d/2}\bar{\varepsilon}_{ox}\cdot\sigma_{xx}dz + \int_{-d/2}^{d/2}(-\bar{\varphi}_{11}\cdot z\cdot\sigma_{xx})dz$$

Since $\bar{\varepsilon}_{ox}$ and $\bar{\varphi}_{11}$ are independent of z, we can take them outside of the through-thickness integrals:

$$\int_{-d/2}^{d/2} \bar{\varepsilon}_{xx}\cdot\sigma_{xx}dz = \bar{\varepsilon}_{ox}\cdot\int_{-d/2}^{d/2}\sigma_{xx}dz + \bar{\varphi}_{11}\cdot\int_{-d/2}^{d/2}(-z\cdot\sigma_{xx})dz = \bar{\varepsilon}_{ox}\cdot\hat{n}_{xx} + \bar{\varphi}_{11}\cdot\hat{m}_{xx} \qquad (14.5.6a)$$

where we have also accounted for Equations (14.2.1a) and (14.2.2a).

In the same fashion, we can express the remaining through-thickness integrals in the left-hand side of Equation (14.5.5) in terms of stress resultants and virtual generalized strains:

$$\int_{-d/2}^{d/2} \bar{\varepsilon}_{yy}\cdot\sigma_{yy}dz = \int_{-d/2}^{d/2}(\bar{\varepsilon}_{oy}-\bar{\varphi}_{22}\cdot z)\cdot\sigma_{yy}dz = \bar{\varepsilon}_{oy}\cdot\int_{-d/2}^{d/2}\sigma_{yy}dz + \bar{\varphi}_{22}\cdot\int_{-d/2}^{d/2}(-z\cdot\sigma_{yy})dz \rightarrow$$

$$\rightarrow \int_{-d/2}^{d/2} \bar{\varepsilon}_{yy}\cdot\sigma_{yy}dz = \bar{\varepsilon}_{oy}\cdot\hat{n}_{yy} + \bar{\varphi}_{22}\cdot\hat{m}_{yy} \qquad (14.5.6b)$$

$$\int_{-d/2}^{d/2} \bar{\gamma}_{xy}\cdot\sigma_{xy}dz = \int_{-d/2}^{d/2}\left(\bar{\gamma}_{oxy}-2\bar{\varphi}_{12}\cdot z\right)\cdot\sigma_{xy}dz = \bar{\gamma}_{oxy}\cdot\int_{-d/2}^{d/2}\sigma_{xy}dz + 2\bar{\varphi}_{12}\cdot\int_{-d/2}^{d/2}(-z\cdot\sigma_{xy})dz \rightarrow$$

$$\rightarrow \int_{-d/2}^{d/2} \bar{\gamma}_{xy}\cdot\sigma_{xy}dz = \bar{\gamma}_{oxy}\cdot\hat{n}_{xy} + 2\bar{\varphi}_{12}\cdot\hat{m}_{xy} \qquad (14.5.6c)$$

$$\int_{-d/2}^{d/2} \bar{\gamma}_{yz} \cdot \sigma_{yz} dz = \bar{\gamma}_{yz} \cdot \int_{-d/2}^{d/2} \sigma_{yz} dz = \bar{\gamma}_{yz} \cdot \hat{q}_y \tag{14.5.6d}$$

$$\int_{-d/2}^{d/2} \bar{\gamma}_{zx} \cdot \sigma_{zx} dz = \bar{\gamma}_{zx} \cdot \int_{-d/2}^{d/2} \sigma_{zx} dz = \bar{\gamma}_{zx} \cdot \hat{q}_x \tag{14.5.6e}$$

Thus, if we account for Equations (14.5.6a–e), the right-hand side of Equation (14.5.5) becomes:

$$\begin{aligned}
\bar{W}_{int} &= \iiint_V \left(\bar{\varepsilon}_{xx} \cdot \sigma_{xx} + \bar{\varepsilon}_{yy} \cdot \sigma_{yy} + \bar{\gamma}_{xy} \cdot \sigma_{xy} + \bar{\gamma}_{yz} \cdot \sigma_{yz} + \bar{\gamma}_{zx} \cdot \sigma_{zx}\right) dV = \\
&= \iint_\Omega \left[\int_{-d/2}^{d/2} \left(\bar{\varepsilon}_{xx} \cdot \sigma_{xx} + \bar{\varepsilon}_{yy} \cdot \sigma_{yy} + \bar{\gamma}_{xy} \cdot \sigma_{xy} + \bar{\gamma}_{yz} \cdot \sigma_{yz} + \bar{\gamma}_{zx} \cdot \sigma_{zx}\right) dz\right] d\Omega = \\
&= \iint_\Omega \left(\bar{\varepsilon}_{ox} \cdot \hat{n}_{xx} + \bar{\varepsilon}_{oy} \cdot \hat{n}_{yy} + \bar{\gamma}_{oxy} \cdot \hat{n}_{xy}\right) d\Omega \\
&+ \iint_\Omega \left(\bar{\varphi}_{11} \cdot \hat{m}_{xx} + \bar{\varphi}_{22} \cdot \hat{m}_{yy} + 2\bar{\varphi}_{12} \cdot \hat{m}_{xy} + \bar{\gamma}_{zx} \cdot \hat{q}_x + \bar{\gamma}_{yz} \cdot \hat{q}_y\right) d\Omega
\end{aligned}$$

Thus, we obtain:

$$\begin{aligned}
\bar{W}_{int} &= \iiint_V \left(\bar{\varepsilon}_{xx} \cdot \sigma_{xx} + \bar{\varepsilon}_{yy} \cdot \sigma_{yy} + \bar{\gamma}_{xy} \cdot \sigma_{xy} + \bar{\gamma}_{yz} \cdot \sigma_{yz} + \bar{\gamma}_{zx} \cdot \sigma_{zx}\right) dV \\
&= \iint_\Omega \left\{\bar{\hat{\varepsilon}}_m\right\}^T \left\{\hat{\sigma}_m\right\} d\Omega + \iint_\Omega \left\{\bar{\hat{\varepsilon}}_b\right\}^T \left\{\hat{\sigma}_b\right\} d\Omega
\end{aligned} \tag{14.5.7}$$

where we have defined the virtual *generalized membrane strain vector*, $\left\{\bar{\hat{\varepsilon}}_m\right\}$:

$$\left\{\bar{\hat{\varepsilon}}_m\right\} = \left[\bar{\varepsilon}_{ox} \quad \bar{\varepsilon}_{oy} \quad \bar{\gamma}_{oxy}\right]^T \tag{14.5.8a}$$

and the *virtual generalized bending (plate) strain vector* $\left\{\bar{\hat{\varepsilon}}_b\right\}$:

$$\left\{\bar{\hat{\varepsilon}}_b\right\} = \left[\bar{\varphi}_{11} \quad \bar{\varphi}_{22} \quad 2\bar{\varphi}_{12} \quad \bar{\gamma}_{zx} \quad \bar{\gamma}_{zy}\right]^T \tag{14.5.8b}$$

The first integral term in Equation (14.5.7) represents the internal virtual work due to membrane behavior and the second integral term represents the internal virtual work due to bending (plate) behavior. We can use the definitions of Equations (14.4.39), (14.4.40), to write Equation (14.5.7) in the following form:

$$\bar{W}_{int} = \iint_\Omega \left\{\bar{\hat{\varepsilon}}\right\}^T \left\{\hat{\sigma}\right\} d\Omega \tag{14.5.9a}$$

If we also account for Equation (14.4.38), which applies for shells made of linearly elastic material, we have:

$$\bar{W}_{int} = \iint_{\Omega} \{\hat{\bar{\varepsilon}}\}^T [\hat{D}] \{\hat{\varepsilon}\} d\Omega \tag{14.5.9b}$$

Remark 14.5.1: For the special case where Equation (14.4.41) applies, we can separate the virtual work terms associated with membrane action, flexure, and shear, and write:

$$\bar{W}_{int} = \iint_{\Omega} \{\hat{\bar{\varepsilon}}_m\}^T [\hat{D}_m] \{\hat{\varepsilon}_m\} d\Omega + \iint_{\Omega} \{\hat{\bar{\varepsilon}}_f\}^T [\hat{D}_f] \{\hat{\varepsilon}_f\} d\Omega + \iint_{\Omega} \{\hat{\bar{\varepsilon}}_s\}^T [\hat{D}_s] \{\hat{\varepsilon}_s\} d\Omega \tag{14.5.9c}$$ ■

Let us now examine the "external work" terms, that is, the right-hand side of the weak form for three-dimensional elasticity, given in Box 9.1.1:

$$\bar{W}_{ext} = \iiint_V (w_x b_x + w_y b_y + w_z b_z) dV + \iint_{\Gamma_{tx}} w_x t_x d\Gamma + \iint_{\Gamma_{ty}} w_y t_y d\Gamma + \iint_{\Gamma_{tz}} w_z t_z d\Gamma \tag{14.5.10}$$

We will once again apply the considerations based on Figure 14.8 for the three-dimensional volume integrals of the body forces:

$$\iiint_V (w_x b_x + w_y b_y + w_z + b_z) dV = \iint_{\Omega} \left[\int_{-d/2}^{d/2} (w_x b_x + w_y b_y + w_z b_z) dz \right] d\Omega$$
$$= \iint_{\Omega} \left[\int_{-d/2}^{d/2} (w_x b_x) dz \right] d\Omega + \iint_{\Omega} \left[\int_{-d/2}^{d/2} (w_y b_y) dz \right] d\Omega + \iint_{\Omega} \left[\int_{-d/2}^{d/2} (w_z b_z) dz \right] d\Omega \tag{14.5.11}$$

The right-hand side of Equation (14.5.11) has three terms. For the first term, we have:

$$\iint_{\Omega} \left[\int_{-d/2}^{d/2} (w_x b_x) dz \right] d\Omega = \iint_{\Omega} \left[\int_{-d/2}^{d/2} (w_{ox} - z \cdot w_{\theta 1}) b_x dz \right] d\Omega$$
$$= \iint_{\Omega} w_{ox} \left[\int_{-d/2}^{d/2} b_x dz \right] d\Omega + \iint_{\Omega} w_{\theta 1} \left[\int_{-d/2}^{d/2} (-z \cdot b_x) dz \right] d\Omega \rightarrow$$
$$\rightarrow \iint_{\Omega} \left[\int_{-d/2}^{d/2} (w_x b_x) dz \right] d\Omega = \iint_{\Omega} w_{ox} p_x d\Omega + \iint_{\Omega} w_{\theta 1} \hat{m}_1 d\Omega \tag{14.5.12}$$

where we have accounted for Equation (14.5.2a) and have tacitly relied on the stipulation that the *distributed membrane force* p_x *per unit shell surface area* is given by integrating the body force over the shell thickness:

$$p_x(x,y) = \int_{-d/2}^{d/2} b_x dz \tag{14.5.13a}$$

and that the *distributed moment* $\hat{m}_1$ *per unit shell surface area* is defined by the following equation.

$$\hat{m}_1(x,y) = \int_{-d/2}^{d/2} (-z \cdot b_x) dz \tag{14.5.13b}$$

Similarly, for the second term of the right-hand side in Equation (14.5.11), we obtain:

$$\iint_\Omega \left[\int_{-d/2}^{d/2} \left(w_y b_y\right) dz \right] d\Omega = \iint_\Omega w_{oy} p_y d\Omega + \iint_\Omega w_{\theta 2} \hat{m}_2 d\Omega \tag{14.5.14}$$

where we used the definitions:

$$p_y(x,y) = \int_{-d/2}^{d/2} b_x dz \tag{14.5.15a}$$

and

$$\hat{m}_2(x,y) = \int_{-d/2}^{d/2} \left(-z \cdot b_y\right) dz \tag{14.5.15b}$$

Finally, the third term in the right-hand side of Equation (14.5.11) yields:

$$\iint_\Omega \left[\int_{-d/2}^{d/2} (w_z b_z) dz \right] d\Omega = \iint_\Omega w_{oz} \left[\int_{-d/2}^{d/2} b_z dz \right] d\Omega = \iint_\Omega w_{oz} p_z d\Omega \tag{14.5.16}$$

where we tacitly defined the *distributed normal force* p_z *per unit shell surface area* as follows.

$$p_z(x,y) = \int_{-d/2}^{d/2} b_z dz \tag{14.5.17}$$

Let us now consider the terms of Equation (14.5.10) giving work from prescribed tractions over natural boundary surface segments, Γ_t. The surface Γ_t can be thought of as a line segment Γ with a thickness equal to d, as shown in Figure 14.9. Just like we converted volume integrals to mid-surface ones, we will transform the boundary surface integrals of a three-dimensional domain to one-dimensional line boundary terms over the edges of the shell mid-surface.

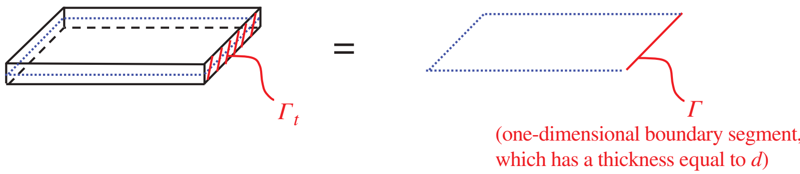


Figure 14.9 Consideration of a three-dimensional boundary surface segment as a one-dimensional boundary segment for the shell mid-surface.

For the first traction boundary term in Equation (14.5.10), we account for Equation (14.5.2a) to obtain:

$$\iint_{\Gamma_t} w_x t_x d\Gamma = \int_{\Gamma} \left(\int_{-d/2}^{d/2} w_x t_x dz \right) dS = \int_{\Gamma} \left[\int_{-d/2}^{d/2} (w_{ox} - z \cdot w_{\theta 1}) t_x dz \right] dS$$

$$= \int_{\Gamma} \left[\int_{-d/2}^{d/2} w_{ox} t_x dz \right] dS + \int_{\Gamma} \left[\int_{-d/2}^{d/2} (-z \cdot w_{\theta 1}) t_x dz \right] dS$$

$$\rightarrow \iint_{\Gamma_t} w_x t_x d\Gamma = \int_{\Gamma} w_{ox} \left[\int_{-d/2}^{d/2} t_x dz \right] dS + \int_{\Gamma} w_{\theta 1} \left[\int_{-d/2}^{d/2} -z \cdot t_x dz \right] dS$$

$$= \int_{\Gamma} w_{ox} \cdot \hat{f}_x dS + \int_{\Gamma} w_{\theta 1} \cdot \bar{\hat{m}}_x dS \tag{14.5.18}$$

where $\hat{f}_x$ is the *distributed membrane force in the x-direction per unit width of the shell boundary line, given by the expression*:

$$\hat{f}_x = \int_{-d/2}^{d/2} t_x dz \tag{14.5.19a}$$

and $\bar{\hat{m}}_x$ is the *prescribed moment in the direction of* θ_1 *per unit width of the shell boundary line*, given by:

$$\bar{\hat{m}}_x = \int_{-d/2}^{d/2} -z \cdot t_x dz \tag{14.5.19b}$$

The second traction boundary term in the right-hand side of Equation (14.5.10) gives:

$$\iint_{\Gamma_t} w_y t_y d\Gamma = \int_{\Gamma}\left(\int_{-d/2}^{d/2} w_y t_y dz\right) dS = \int_{\Gamma}\left[\int_{-d/2}^{d/2} \left(w_{oy} - z\cdot w_{\theta 2}\right) t_y dz\right] dS$$

$$= \int_{\Gamma}\left[\int_{-d/2}^{d/2} w_{oy} t_y dz\right] dS + \int_{\Gamma}\left[\int_{-d/2}^{d/2} (-z\cdot w_{\theta 2}) t_y dz\right] dS$$

$$\rightarrow \iint_{\Gamma_t} w_y t_y d\Gamma = \int_{\Gamma} w_{oy}\left[\int_{-d/2}^{d/2} t_y dz\right] dS + \int_{\Gamma} w_{\theta 2}\left[\int_{-d/2}^{d/2} -z\cdot t_y dz\right] dS$$

$$= \int_{\Gamma} w_{oy}\cdot \hat{f}_y dS + \int_{\Gamma} w_{\theta 2}\cdot \bar{m}_y dS \tag{14.5.20}$$

where $\hat{f}_y$ and $\bar{m}_y$ are the *distributed prescribed membrane force in the y-direction and moment in the direction of θ_2 per unit width of the shell boundary*, given by the expressions:

$$\hat{f}_y = \int_{-d/2}^{d/2} t_y\, dz \tag{14.5.21a}$$

and

$$\bar{m}_y = \int_{-d/2}^{d/2} -z\cdot t_y\, dz \tag{14.5.21b}$$

Finally, the third traction boundary term in the right-hand side of Equation (14.5.10) yields:

$$\iint_{\Gamma_t} w_z t_z d\Gamma = \int_{\Gamma}\left[\int_{-d/2}^{d/2} w_{oz} t_z dz\right] dS = \int_{\Gamma} w_{oz}\left[\int_{-d/2}^{d/2} t_z dz\right] dS = \int_{\Gamma} w_{oz}\cdot \hat{f}_z dS \tag{14.5.22}$$

where the *distributed prescribed shear force $\hat{f}_z$ in the z-direction, per unit width of the shell boundary edge, is given by the expression*:

$$\hat{f}_z = \int_{-d/2}^{d/2} t_z dz \tag{14.5.23}$$

If we plug Equations (14.5.9b), (14.5.12), (14.5.14), (14.5.16), and (14.5.18), (14.5.20), and (14.5.22) into the weak form for three-dimensional elasticity (Box 9.1.1), and account for the shell essential boundary conditions, we eventually obtain the *weak form for shell problems*, which is the principle of virtual work for such structures and is provided in Box 14.5.1.

Box 14.5.1 Weak Form for Shell Problems

$$\iint_{\Omega} \{\hat{\bar{\varepsilon}}\}^T [\hat{D}] \{\hat{\varepsilon}\} d\Omega = \iint_{\Omega} \{w\}^T \{p\} dV + \int_{\Gamma_{tx}} w_{ox} \hat{f}_x dS + \int_{\Gamma_{ty}} w_{oy} \hat{f}_y dS + \int_{\Gamma_{tz}} w_{oz} \hat{f}_z dS$$

$$+ \int_{\Gamma_{t\theta 1}} w_{\theta 1} \bar{\hat{m}}_x dS + \int_{\Gamma_{t\theta 2}} w_{\theta 2} \bar{\hat{m}}_y dS$$

where $\{p\} = [p_x \ \ p_y \ \ p_z \ \ \hat{m}_1 \ \ \hat{m}_2]^T$

$\forall \ \{w\} = [w_{ox} \ \ w_{ox} \ \ w_{ox} \ \ w_{\theta 1} \ \ w_{\theta 2}]^T$

with $w_x = 0$ on Γ_{ux}, $w_y = 0$ on Γ_{uy}, $w_z = 0$ on Γ_{uz}, $w_{\theta 1} = 0$ on $\Gamma_{u\theta 1}$, $w_{\theta 2} = 0$ at $\Gamma_{u\theta 2}$

$u_{ox} = \bar{u}_{ox}$ on Γ_{ux}

$u_{oy} = \bar{u}_{oy}$ on Γ_{uy}

$u_{oz} = \bar{u}_{oz}$ on Γ_{uz}

$\theta_1 = \bar{\theta}_1$ on $\Gamma_{u\theta 1}$

$\theta_2 = \bar{\theta}_2$ on $\Gamma_{u\theta 2}$

14.6 Finite Element Formulation for Shell Structures

We will now move on to examine the finite element analysis of shell structures. Once again, we assume that we subdivide the structure into N_e subdomains called the elements, and also that we have a piecewise approximation for the kinematic quantities of the reference surface in each element e. We will also assume that each element has n nodes, with six degrees of freedom per node (three translations and three rotations), that the shell surface lies in the xy plane, and that we use an isoparametric formulation, that is, the shape functions are polynomials with respect to parametric coordinates ξ and η (in accordance with Section 6.4). The finite element approximation in each element e is defined by the following expressions.

$$u_{ox}^{(e)}(\xi,\eta) = \sum_{i=1}^{n} \left(N_i^{(e)}(\xi,\eta) \cdot u_{xi}^{(e)} \right) \tag{14.6.1a}$$

$$u_{oy}^{(e)}(\xi,\eta) = \sum_{i=1}^{n} \left(N_i^{(e)}(\xi,\eta) \cdot u_{yi}^{(e)} \right) \tag{14.6.1b}$$

$$u_{oz}^{(e)}(\xi,\eta) = \sum_{i=1}^{n} \left(N_i^{(e)}(\xi,\eta) \cdot u_{zi}^{(e)} \right) \tag{14.6.1c}$$

$$\theta_1^{(e)}(\xi,\eta) = -\sum_{i=1}^{n} \left(N_i^{(e)}(\xi,\eta) \cdot \theta_{yi}^{(e)} \right) \tag{14.6.1d}$$

$$\theta_2^{(e)}(\xi,\eta) = \sum_{i=1}^{n} \left(N_i^{(e)}(\xi,\eta) \cdot \theta_{xi}^{(e)} \right) \tag{14.6.1e}$$

We can cast Equations (14.6.1a–e) in matrix form, as shown in Figure 14.10.

$$\begin{Bmatrix} u_{xo}(\xi,\eta) \\ u_{yo}(\xi,\eta) \\ u_{zo}(\xi,\eta) \\ \theta_1(\xi,\eta) \\ \theta_2(\xi,\eta) \end{Bmatrix} = \left[\overbrace{\begin{matrix} N_1^{(4Q)}(\xi,\eta) & 0 & 0 & 0 & 0 \\ 0 & N_1^{(4Q)}(\xi,\eta) & 0 & 0 & 0 \\ 0 & 0 & N_1^{(4Q)}(\xi,\eta) & 0 & 0 \\ 0 & 0 & 0 & -N_1^{(4Q)}(\xi,\eta) & 0 \\ 0 & 0 & 0 & 0 & N_1^{(4Q)}(\xi,\eta) \end{matrix}}^{\text{Shape functions for node 1}} \; \overbrace{\begin{matrix} N_2^{(4Q)}(\xi,\eta) & 0 & 0 & 0 & 0 \\ 0 & N_2^{(4Q)}(\xi,\eta) & 0 & 0 & 0 \\ 0 & 0 & N_2^{(4Q)}(\xi,\eta) & 0 & 0 \\ 0 & 0 & 0 & -N_2^{(4Q)}(\xi,\eta) & 0 \\ 0 & 0 & 0 & 0 & N_2^{(4Q)}(\xi,\eta) \end{matrix}}^{\text{Shape functions for node 2}} \; \begin{matrix} \ldots \\ \ldots \\ \ldots \\ \ldots \\ \ldots \end{matrix}^{\ldots\ldots} \; \overbrace{\begin{matrix} N_n^{(4Q)}(\xi,\eta) & 0 & 0 & 0 & 0 \\ 0 & N_n^{(4Q)}(\xi,\eta) & 0 & 0 & 0 \\ 0 & 0 & N_n^{(4Q)}(\xi,\eta) & 0 & 0 \\ 0 & 0 & 0 & -N_n^{(4Q)}(\xi,\eta) & 0 \\ 0 & 0 & 0 & 0 & N_n^{(4Q)}(\xi,\eta) \end{matrix}}^{\text{Shape functions for node 'n'}} \right] \begin{Bmatrix} u_{x1}^{(e)} \\ u_{y1}^{(e)} \\ u_{z1}^{(e)} \\ \theta_{x1}^{(e)} \\ \theta_{y1}^{(e)} \\ u_{x2}^{(e)} \\ u_{y2}^{(e)} \\ u_{z2}^{(e)} \\ \theta_{x2}^{(e)} \\ \theta_{y2}^{(e)} \\ \vdots \\ u_{xn}^{(e)} \\ u_{yn}^{(e)} \\ u_{zn}^{(e)} \\ \theta_{xn}^{(e)} \\ \theta_{yn}^{(e)} \end{Bmatrix}$$

Figure 14.10 Matrix expression giving the approximate displacement field for a shell element.

We can also write a block matrix form for the finite element approximation. We can write:

$$\begin{Bmatrix} u_{ox}^{(e)}(\xi,\eta) \\ u_{oy}^{(e)}(\xi,\eta) \\ u_{oz}^{(e)}(\xi,\eta) \\ \theta_1^{(e)}(\xi,\eta) \\ \theta_2^{(e)}(\xi,\eta) \end{Bmatrix} = \left[N_1^{(e)}(\xi,\eta) \right] \left\{ U_1^{(e)} \right\} + \left[N_2^{(e)}(\xi,\eta) \right] \left\{ U_2^{(e)} \right\} + \ldots + \left[N_n^{(e)}(\xi,\eta) \right] \left\{ U_n^{(e)} \right\} \rightarrow$$

$$\rightarrow \begin{Bmatrix} u_{ox}^{(e)}(\xi,\eta) \\ u_{oy}^{(e)}(\xi,\eta) \\ u_{oz}^{(e)}(\xi,\eta) \\ \theta_1^{(e)}(\xi,\eta) \\ \theta_2^{(e)}(\xi,\eta) \end{Bmatrix} = \left[\left[N_1^{(e)}(\xi,\eta) \right] \left[N_2^{(e)}(\xi,\eta) \right] \cdots \left[N_n^{(e)}(\xi,\eta) \right] \right] \begin{Bmatrix} \left\{ U_1^{(e)} \right\} \\ \left\{ U_2^{(e)} \right\} \\ \vdots \\ \left\{ U_n^{(e)} \right\} \end{Bmatrix} \tag{14.6.2}$$

where $\left[N_i^{(e)}(\xi,\eta) \right]$ is a (5×6) array, with the shape functions expressing the contribution of node i to the approximate fields:

$$\left[N_i^{(e)}(\xi,\eta) \right] = \begin{bmatrix} N_i^{(e)}(\xi,\eta) & 0 & 0 & 0 & 0 & 0 \\ 0 & N_i^{(e)}(\xi,\eta) & 0 & 0 & 0 & 0 \\ 0 & 0 & N_i^{(e)}(\xi,\eta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -N_i^{(e)}(\xi,\eta) & 0 \\ 0 & 0 & 0 & N_i^{(e)}(\xi,\eta) & 0 & 0 \end{bmatrix} \tag{14.6.3}$$

and $\left\{U_i^{(e)}\right\}$ is a vector with the six degrees of freedom of nodal point i:

$$\left\{U_i^{(e)}\right\} = \left[u_{xi}^{(e)} \; u_{yi}^{(e)} \; u_{zi}^{(e)} \; \theta_{xi}^{(e)} \; \theta_{yi}^{(e)} \; \theta_{zi}^{(e)}\right]^T \tag{14.6.4}$$

In our derivations, we will also use the more concise equation:

$$\left\{u^{(e)}\right\} = \left[N^{(e)}\right]\left\{U^{(e)}\right\} \tag{14.6.5}$$

where

$$\left[N^{(e)}\right] = \left[\left[N_1^{(e)}(\xi,\eta)\right] \; \left[N_2^{(e)}(\xi,\eta)\right] \; \cdots \; \left[N_n^{(e)}(\xi,\eta)\right]\right] \tag{14.6.6}$$

$$\left\{u^{(e)}\right\} = \left[u_{ox}^{(e)}(\xi,\eta) \; u_{oy}^{(e)}(\xi,\eta) \; u_{oz}^{(e)}(\xi,\eta) \; \theta_1^{(e)}(\xi,\eta) \; \theta_2^{(e)}(\xi,\eta)\right]^T \tag{14.6.7}$$

and

$$\left\{U^{(e)}\right\} = \begin{Bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \vdots \\ \left\{U_n^{(e)}\right\} \end{Bmatrix} \tag{14.6.8}$$

Remark 14.6.1: In Equations (14.6.2) and (14.6.8), we have included the nodal rotations $\theta_{zi}^{(e)}$ along the z-axis, for each nodal point i. Although these rotations do not affect any of the approximate fields for a flat shell, we will still retain them for consistency (usually, three-dimensional shell models include six degrees of freedom at each node) and also to facilitate transition to discussion of nonplanar shell formulations, where the kinematics may depend on all rotational degrees of freedom of the nodal points! ∎

Remark 14.6.2: It is important to notice the negative (–) sign on the shape functions multiplying the nodal rotations in the y-direction. This sign stems from the positive sign convention for $\theta_1^{(e)}(\xi,\eta)$. ∎

The generalized strain vectors for a shell element are given by appropriate differentiations of the kinematic fields of the mid-surface. Specifically, we can establish expressions for the approximate membrane generalized strains:

$$\left\{\hat{\varepsilon}_m^{(e)}\right\} = \begin{Bmatrix} \varepsilon_{ox}^{(e)} \\ \varepsilon_{oy}^{(e)} \\ \gamma_{oxy}^{(e)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_{xo}^{(e)}}{\partial x} \\ \dfrac{\partial u_{yo}^{(e)}}{\partial y} \\ \dfrac{\partial u_{xo}^{(e)}}{\partial y} + \dfrac{\partial u_{yo}^{(e)}}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial x}\displaystyle\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot u_{xi}^{(e)}\right) \\ \dfrac{\partial}{\partial y}\displaystyle\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot u_{yi}^{(e)}\right) \\ \dfrac{\partial}{\partial y}\displaystyle\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot u_{xi}^{(e)}\right) + \dfrac{\partial}{\partial x}\displaystyle\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot u_{yi}^{(e)}\right) \end{Bmatrix} \tag{14.6.8a}$$

the approximate bending (flexure) generalized strains:

$$\left\{\hat{\varepsilon}_f^{(e)}\right\} = \begin{Bmatrix} \varphi_{11}^{(e)} \\ \varphi_{22}^{(e)} \\ 2\varphi_{12}^{(e)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial\theta_1^{(e)}}{\partial x} \\ \dfrac{\partial\theta_2^{(e)}}{\partial y} \\ \dfrac{\partial\theta_1^{(e)}}{\partial y} + \dfrac{\partial u_2^{(e)}}{\partial x} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial x}\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot\theta_{yi}^{(e)}\right) \\ \dfrac{\partial}{\partial y}\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot\theta_{xi}^{(e)}\right) \\ \dfrac{\partial}{\partial y}\sum_{i=1}^{n}\left(-N_i^{(e)}(\xi,\eta)\cdot\theta_{yi}^{(e)}\right) + \dfrac{\partial}{\partial x}\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot\theta_{xi}^{(e)}\right) \end{Bmatrix}$$

(14.6.8b)

and the approximate shear generalized strains:

$$\left\{\hat{\varepsilon}_s^{(e)}\right\} = \begin{Bmatrix} \gamma_{zx}^{(e)} \\ \gamma_{zy}^{(e)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_{oz}^{(e)}}{\partial x} - \theta_1^{(e)} \\ \dfrac{\partial u_{oz}^{(e)}}{\partial x} - \theta_2^{(e)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial}{\partial x}\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot u_{zi}^{(e)}\right) - \sum_{i=1}^{n}\left(-N_i^{(e)}(\xi,\eta)\cdot\theta_{yi}^{(e)}\right) \\ \dfrac{\partial}{\partial x}\sum_{i=1}^{n}\left(N_i^{(e)}(\xi,\eta)\cdot u_{zi}^{(e)}\right) - \sum_{i=1}^{n}\left(-N_i^{(e)}(\xi,\eta)\cdot\theta_{xi}^{(e)}\right) \end{Bmatrix}$$

(14.6.8c)

To obtain finite element equations, we will combine Equations (14.6.8a–c) in a block matrix form, where a matrix premultiplies the vector $\{U^{(e)}\}$ defined by Equation (14.6.7). First we write three distinct expressions for the membrane, bending, and shear strains:

$$\left\{\hat{\varepsilon}_m^{(e)}\right\} = \left[B_m^{(e)}(\xi,\eta)\right]\left\{U^{(e)}\right\} = \left[\left[B_{m1}^{(e)}(\xi,\eta)\right] \quad \left[B_{m2}^{(e)}(\xi,\eta)\right] \quad \cdots \quad \left[B_{mn}^{(e)}(\xi,\eta)\right]\right]\begin{Bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \vdots \\ \left\{U_n^{(e)}\right\} \end{Bmatrix}$$

(14.6.9a)

$$\left\{\hat{\varepsilon}_f^{(e)}\right\} = \left[B_f^{(e)}(\xi,\eta)\right]\left\{U^{(e)}\right\} = \left[\left[B_{f1}^{(e)}(\xi,\eta)\right] \quad \left[B_{f2}^{(e)}(\xi,\eta)\right] \quad \cdots \quad \left[B_{fn}^{(e)}(\xi,\eta)\right]\right]\begin{Bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \vdots \\ \left\{U_n^{(e)}\right\} \end{Bmatrix}$$

(14.6.9b)

and

$$\left\{\hat{\varepsilon}_s^{(e)}\right\} = \left[B_s^{(e)}(\xi,\eta)\right]\left\{U^{(e)}\right\} = \left[\left[B_{s1}^{(e)}(\xi,\eta)\right] \quad \left[B_{s2}^{(e)}(\xi,\eta)\right] \quad \cdots \quad \left[B_{sn}^{(e)}(\xi,\eta)\right]\right]\begin{Bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \vdots \\ \left\{U_n^{(e)}\right\} \end{Bmatrix} \tag{14.6.9c}$$

The matrices $\left[B_{mi}^{(e)}(\xi,\eta)\right]$, $\left[B_{fi}^{(e)}(\xi,\eta)\right]$ and $\left[B_{si}^{(e)}(\xi,\eta)\right]$ in Equations (14.6.9a–c) are the block strain-displacement arrays for nodal point *i*, corresponding to membrane strains, flexural strains, and shear strains, respectively. These arrays are given by the following equations.

$$\left[B_{mi}^{(e)}(\xi,\eta)\right] = \begin{bmatrix} \frac{\partial N_i^{(e)}}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_i^{(e)}}{\partial y} & 0 & 0 & 0 & 0 \\ \frac{\partial N_i^{(e)}}{\partial y} & \frac{\partial N_i^{(e)}}{\partial x} & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14.6.10a}$$

$$\left[B_{fi}^{(e)}(\xi,\eta)\right] = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{\partial N_i^{(e)}}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial N_i^{(e)}}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_i^{(e)}}{\partial x} & -\frac{\partial N_i^{(e)}}{\partial y} & 0 \end{bmatrix} \tag{14.6.10b}$$

$$\left[B_{si}^{(e)}(\xi,\eta)\right] = \begin{bmatrix} 0 & 0 & \frac{\partial N_i^{(e)}}{\partial x} & 0 & N_i^{(e)} & 0 \\ 0 & 0 & \frac{\partial N_i^{(e)}}{\partial y} & -N_i^{(e)} & 0 & 0 \end{bmatrix} \tag{14.6.10c}$$

We can also combine Equations (14.6.9a–c) in a unique, block matrix expression:

$$\left\{\hat{\varepsilon}^{(e)}\right\} = \begin{Bmatrix} \left\{\hat{\varepsilon}_m^{(e)}\right\} \\ \left\{\hat{\varepsilon}_f^{(e)}\right\} \\ \left\{\hat{\varepsilon}_s^{(e)}\right\} \end{Bmatrix} = \left[\begin{bmatrix} \left[B_{m1}^{(e)}(\xi,\eta)\right] \\ \left[B_{f1}^{(e)}(\xi,\eta)\right] \\ \left[B_{s1}^{(e)}(\xi,\eta)\right] \end{bmatrix} \begin{bmatrix} \left[B_{m2}^{(e)}(\xi,\eta)\right] \\ \left[B_{f2}^{(e)}(\xi,\eta)\right] \\ \left[B_{s2}^{(e)}(\xi,\eta)\right] \end{bmatrix} \cdots \begin{bmatrix} \left[B_{mn}^{(e)}(\xi,\eta)\right] \\ \left[B_{fn}^{(e)}(\xi,\eta)\right] \\ \left[B_{sn}^{(e)}(\xi,\eta)\right] \end{bmatrix}\right] \begin{Bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \vdots \\ \left\{U_n^{(e)}\right\} \end{Bmatrix} \rightarrow$$

$$\rightarrow \left\{\hat{\varepsilon}^{(e)}\right\} = \left[B^{(e)}(\xi,\eta)\right]\left\{U^{(e)}\right\} \tag{14.6.10d}$$

Equation (14.6.10d) establishes the generalized strain-displacement matrix $[B^{(e)}]$ for the shell element e.

Now, all the integral terms in the weak form (Box 14.5.1) can be separated into the element contributions. We also assume the same approximation for the virtual displacement/rotation fields as for the actual displacement/rotation fields. Thus, the virtual generalized strain vector can be written as:

$$\left\{\bar{\hat{\varepsilon}}^{(e)}\right\}^T = \left\{W^{(e)}\right\}^T \left[B^{(e)}\right]^T \tag{14.6.11}$$

where $\{W^{(e)}\}$ is a virtual nodal displacement/rotation vector for element e. The left-hand side of the weak form can be written as a sum of the individual element contributions:

$$\iint_{\Omega} \{\bar{\hat{\varepsilon}}\}^T [\hat{D}] \{\hat{\varepsilon}\} dV = \sum_{e=1}^{N_e} \left(\iint_{\Omega^{(e)}} \{\bar{\hat{\varepsilon}}\}^T [\hat{D}] \{\hat{\varepsilon}\} dV \right)$$

If we also account for the gather operation of each element (per Section B.2), that is, $\left\{U^{(e)}\right\} = \left[L^{(e)}\right]^T\{U\}$ and $\left\{W^{(e)}\right\}^T = \{W\}^T\left[L^{(e)}\right]^T$, we have:

$$\iint_{\Omega} \{\bar{\hat{\varepsilon}}\}^T [\hat{D}] \{\hat{\varepsilon}\} d\Omega = \sum_{e=1}^{N_e} \left(\iint_{\Omega^{(e)}} \left(\{W\}^T \left[L^{(e)}\right]^T \left[B^{(e)}\right]^T \left[\hat{D}^{(e)}\right] \left[B^{(e)}\right] \left[L^{(e)}\right] \{U\} \right) dV \right)$$

$$= \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \iint_{\Omega^{(e)}} \left[B^{(e)}\right]^T \left[\hat{D}^{(e)}\right] \left[B^{(e)}\right] dV \left[L^{(e)}\right] \right) \{U\}$$

$$\rightarrow \iint_{\Omega} \{\bar{\hat{\varepsilon}}\}^T [\hat{D}] \{\hat{\varepsilon}\} d\Omega = \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \left[k^{(e)}\right] \left[L^{(e)}\right] \right) \{U\} \tag{14.6.12}$$

where $[k^{(e)}]$ is the *stiffness matrix for element e*, given by:

$$\left[k^{(e)}\right] = \iint_{\Omega^{(e)}} \left[B^{(e)}\right]^T \left[\hat{D}^{(e)}\right] \left[B^{(e)}\right] dV \tag{14.6.13}$$

The external virtual work (the right-hand side of the weak form in Box 14.5.1) becomes:

$$\begin{aligned}
&\iint_{\Omega} \{w\}^T \{p\} dV + \int_{\Gamma_{tx}} w_{ox}\hat{f}_x dS + \int_{\Gamma_{ty}} w_{oy}\hat{f}_y dS + \int_{\Gamma_{tz}} w_{oz}\hat{f}_z dS + \int_{\Gamma_{t\theta 1}} w_{\theta 1}\bar{\hat{m}}_x dS + \int_{\Gamma_{t\theta 2}} w_{\theta 2}\bar{\hat{m}}_y dS \\
&= \iint_{\Omega} \{w\}^T \{p\} d\Omega + \int_{\Gamma_{tx}} \{w\}^T \left\{\hat{f}_x\right\} dS + \int_{\Gamma_{ty}} \{w\}^T \left\{\hat{f}_y\right\} dS + \int_{\Gamma_{tz}} \{w\}^T \left\{\hat{f}_z\right\} dS \\
&+ \int_{\Gamma_{t\theta 1}} \{w\}^T \{\hat{m}_x\} dS + \int_{\Gamma_{t\theta 2}} \{w\}^T \{\hat{m}_y\} dS
\end{aligned} \tag{14.6.14}$$

where we have defined the following expanded vectors:

$$\left\{\hat{f}_x\right\} = \left[\hat{f}_x \;\; 0 \;\; 0 \;\; 0 \;\; 0\right]^T \tag{14.6.15a}$$

$$\left\{\hat{f}_y\right\} = \left[0 \;\; \hat{f}_y \;\; 0 \;\; 0 \;\; 0\right]^T \tag{14.6.15b}$$

$$\left\{\hat{f}_y\right\} = \left[0 \;\; 0 \;\; \hat{f}_z \;\; 0 \;\; 0\right]^T \tag{14.6.15c}$$

$$\{\hat{m}_x\} = \left[0 \;\; 0 \;\; 0 \;\; \bar{\hat{m}}_x \;\; 0\right]^T \tag{14.6.15d}$$

$$\{\hat{m}_y\} = \left[0 \;\; 0 \;\; 0 \;\; \bar{\hat{m}}_y \;\; 0\right]^T \tag{14.6.15e}$$

Now, we take the first integral term in the right-hand side of Equation (14.6.14), and we separate it into individual element contributions. We also account for the fact that the virtual displacement field in each element is approximated with the same approximation (the same shape functions) as the actual displacement field:

$$\left\{w^{(e)}\right\}^T = \left[w_{ox}^{(e)}(\xi,\eta) \;\; w_{oy}^{(e)}(\xi,\eta) \;\; w_{oz}^{(e)}(\xi,\eta) \;\; w_{\theta 1}^{(e)}(\xi,\eta) \;\; w_{\theta 2}^{(e)}(\xi,\eta)\right] = \left\{W^{(e)}\right\}^T \left[N^{(e)}\right]^T \tag{14.6.16}$$

We now have:

$$\begin{aligned}
&\iint_{\Omega} \{w\}^T \{p\} dV = \sum_{e=1}^{N_e} \left(\iint_{\Omega^{(e)}} \{w\}^T \{p\} dV \right) = \sum_{e=1}^{N_e} \left(\iint_{\Omega^{(e)}} \{W\}^T \left[L^{(e)}\right]^T \left[N^{(e)}\right]^T \{p\} dV \right) \\
&= \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \iint_{\Omega^{(e)}} \left[N^{(e)}\right]^T \{p\} dV \right) \rightarrow \\
&\rightarrow \iint_{\Omega} \{w\}^T \{p\} dV = \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \left\{f_{\Omega}^{(e)}\right\} \right)
\end{aligned} \tag{14.6.17}$$

where

$$\left\{f_{\Omega}^{(e)}\right\} = \iint_{\Omega^{(e)}} \left[N^{(e)}\right]^T \{p\} dV \quad (14.6.18)$$

The first natural boundary integral term in the right-hand-side of Equation (14.6.14) becomes:

$$\int_{\Gamma_{tx}} \{w\}^T \left\{\hat{f}_x\right\} dS = \sum_{e=1}^{N_e} \left(\int_{\Gamma_{tx}^{(e)}} \{w\}^T \left\{\hat{f}_x\right\} dS \right) = \sum_{e=1}^{N_e} \left(\int_{\Gamma_{tx}^{(e)}} \{W\}^T \left[L^{(e)}\right]^T \left[N^{(e)}\right] \left\{\hat{f}_x\right\} dS \right) \rightarrow$$

$$\rightarrow \int_{\Gamma_{tx}} \{w\}^T \left\{\hat{f}_x\right\} dS = \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right] \int_{\Gamma_{tx}^{(e)}} \left[N^{(e)}\right]^T \left\{\hat{f}_x\right\} dS \right) \quad (14.6.19a)$$

In the same fashion:

$$\int_{\Gamma_{ty}} \{w\}^T \left\{\hat{f}_y\right\} dS = \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \int_{\Gamma_{ty}^{(e)}} \left[N^{(e)}\right]^T \left\{\hat{f}_y\right\} dS \right) \quad (14.6.19b)$$

$$\int_{\Gamma_{tz}} \{w\}^T \left\{\hat{f}_z\right\} dS = \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \int_{\Gamma_{tz}^{(e)}} \left[N^{(e)}\right]^T \left\{\hat{f}_y\right\} dS \right) \quad (14.6.19c)$$

$$\int_{\Gamma_{t\theta 1}} \{w\}^T \{\hat{m}_x\} dS = \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \int_{\Gamma_{t\theta 1}^{(e)}} \left[N^{(e)}\right]^T \{\hat{m}_x\} dS \right) \quad (14.6.19d)$$

$$\int_{\Gamma_{t\theta 2}} \{w\}^T \{\hat{m}_y\} dS = \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \int_{\Gamma_{t\theta 2}^{(e)}} \left[N^{(e)}\right]^T \{\hat{m}_y\} dS \right) \quad (14.6.19e)$$

Plugging Equations (14.6.18) and (14.6.19) into (14.6.14) yields:

$$\iint_{\Omega} \{w\}^T \{p\} dV + \int_{\Gamma_{tx}} \{w\}^T \left\{\hat{f}_x\right\} dS + \int_{\Gamma_{ty}} \{w\}^T \left\{\hat{f}_y\right\} dS + \int_{\Gamma_{tz}} \{w\}^T \left\{\hat{f}_z\right\} dS$$

$$+ \int_{\Gamma_{t\theta 1}} \{w\}^T \{\hat{m}_x\} dS + \int_{\Gamma_{t\theta 2}} \{w\}^T \{\hat{m}_y\} dS$$

$$= \{W\}^T \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \left\{f^{(e)}\right\} \right) \quad (14.6.20)$$

where $\{f^{(e)}\}$ is the element's *equivalent nodal force/moment vector*:

$$\left\{f^{(e)}\right\} = \left\{f_{\Omega}^{(e)}\right\} + \left\{f_{\Gamma}^{(e)}\right\} \tag{14.6.21}$$

and we have also defined a vector $\left\{f_{\Gamma}^{(e)}\right\}$, containing the equivalent nodal forces/moments due to natural boundary conditions:

$$\left\{f_{\Gamma}^{(e)}\right\} = \int_{\Gamma_{tx}^{(e)}} \left[N^{(e)}\right]^T \left\{\hat{f}_x\right\} dS + \int_{\Gamma_{ty}^{(e)}} \left[N^{(e)}\right]^T \left\{\hat{f}_y\right\} dS + \int_{\Gamma_{tz}^{(e)}} \left[N^{(e)}\right]^T \left\{\hat{f}_z\right\} dS + \int_{\Gamma_{t\theta1}^{(e)}} \left[N^{(e)}\right]^T \{\hat{m}_x\} dS + \int_{\Gamma_{t\theta2}^{(e)}} \left[N^{(e)}\right]^T \{\hat{m}_y\} dS \tag{14.6.22}$$

Accounting for Equations (14.6.12) and (14.6.20) allows us to write the finite element approximation of the weak form:

$$\{W\}^T[K]\{U\} = \{W\}^T\{f\} \rightarrow \{W\}^T([K]\{U\} - \{f\}) = 0 \tag{14.6.23}$$

where $[K]$ is the *global stiffness matrix*:

$$[K] = \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \left[k^{(e)}\right]\left[L^{(e)}\right]\right) \tag{14.6.24}$$

and $\{f\}$ is the *global equivalent nodal force vector*:

$$\{f\} = \sum_{e=1}^{N_e} \left(\left[L^{(e)}\right]^T \left\{f^{(e)}\right\}\right) \tag{14.6.25}$$

Since Equation (14.6.23) must apply for all arbitrary global virtual nodal displacement/rotation vectors, $\{W\}$, the only way to always obtain a zero right-hand side in the equation is to have:

$$[K]\{U\} - \{f\} = \{0\} \tag{14.6.26}$$

Equation (14.6.26) constitutes the *global (structural) finite element equations* for a shell structure. Once again, this equation is mathematically identical to that obtained for the finite element solution of other problems in previous chapters. In the following section, we will specifically examine the case of a four-node, quadrilateral shell element.

14.7 Four-Node Planar (Flat) Shell Finite Element

One of the simplest cases of shell finite elements is the four-node planar shell element, shown in Figure 14.11. For this element, we can use an isoparametric formulation and employ the same shape functions that we used for the four-node quadrilateral (4Q) element in Section 6.4. The nodal degrees of freedom for a four-node, three-dimensional, flat shell element are shown in Figure 14.11. It may be worth emphasizing that the

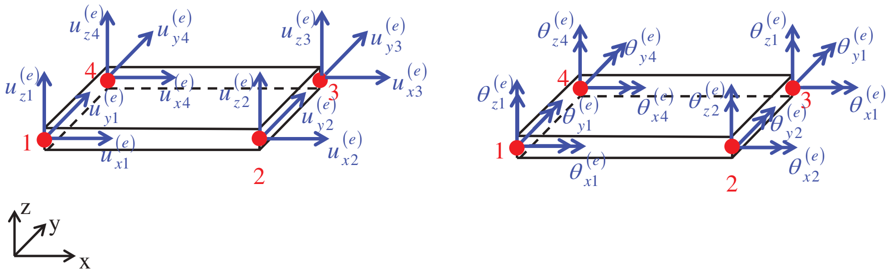


Figure 14.11 Four-node flat shell finite element and nodal degrees of freedom.

formulation that we will present is valid for a general quadrilateral flat shell element (i.e., the element does not need to be rectangular as that of Figure 14.11).

The approximation for the displacement fields is a special case of Equation (14.6.2), if we set $n = 4$:

$$\begin{Bmatrix} u_{ox}^{(e)}(\xi,\eta) \\ u_{oy}^{(e)}(\xi,\eta) \\ u_{oz}^{(e)}(\xi,\eta) \\ \theta_1^{(e)}(\xi,\eta) \\ \theta_2^{(e)}(\xi,\eta) \end{Bmatrix} = \left[\left[N_1^{(4Q)}(\xi,\eta)\right] \quad \left[N_2^{(4Q)}(\xi,\eta)\right] \quad \left[N_3^{(4Q)}(\xi,\eta)\right] \quad \left[N_n^{(4Q)}(\xi,\eta)\right]\right] \begin{Bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \left\{U_3^{(e)}\right\} \\ \left\{U_4^{(e)}\right\} \end{Bmatrix} \tag{14.7.1}$$

Since the formulation is isoparametric, we can also establish a parametric representation of the geometry, through a coordinate mapping from the parametric domain to the physical domain:

$$x(\xi,\eta) = N_1^{(4Q)}(\xi,\eta)\cdot x_1^{(e)} + N_2^{(4Q)}(\xi,\eta)\cdot x_2^{(e)} + N_3^{(4Q)}(\xi,\eta)\cdot x_3^{(e)} + N_4^{(4Q)}(\xi,\eta)\cdot x_4^{(e)} \tag{14.7.2a}$$

$$y(\xi,\eta) = N_1^{(4Q)}(\xi,\eta)\cdot y_1^{(e)} + N_2^{(4Q)}(\xi,\eta)\cdot y_2^{(e)} + N_3^{(4Q)}(\xi,\eta)\cdot y_3^{(e)} + N_4^{(4Q)}(\xi,\eta)\cdot y_4^{(e)} \tag{14.7.2b}$$

Equations (14.7.2a,b) can also be written as follows:

$$x(\xi,\eta) = \left[N^{(4Q)}(\xi,\eta)\right]\left\{x^{(e)}\right\} \tag{14.7.3a}$$

$$y(\xi,\eta) = \left[N^{(4Q)}(\xi,\eta)\right]\left\{y^{(e)}\right\} \tag{14.7.3b}$$

where

$$\left\{x^{(e)}\right\} = \left[x_1^{(e)} \quad x_2^{(e)} \quad x_3^{(e)} \quad x_4^{(e)}\right]^T \tag{14.7.4a}$$

$$\left\{y^{(e)}\right\} = \left[y_1^{(e)} \quad y_2^{(e)} \quad y_3^{(e)} \quad y_4^{(e)}\right]^T \tag{14.7.4b}$$

and

$$\left[N^{(4Q)}(\xi,\eta)\right] = \left[N_1^{(4Q)}(\xi,\eta) \quad N_2^{(4Q)}(\xi,\eta) \quad N_3^{(4Q)}(\xi,\eta) \quad N_4^{(4Q)}(\xi,\eta)\right] \tag{14.7.4c}$$

It is reminded that the shape functions of a 4Q element are:

$$N_1^{(4Q)}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \tag{14.7.5a}$$

$$N_2^{(4Q)}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \tag{14.7.5b}$$

$$N_3^{(4Q)}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \tag{14.7.5c}$$

$$N_4^{(4Q)}(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta) \tag{14.7.5d}$$

We also need to establish the Jacobian matrix $[J]$ of the coordinate mapping:

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4}\left(\frac{\partial N_i^{(4Q)}}{\partial \xi}\cdot x_i^{(e)}\right) & \sum_{i=1}^{4}\left(\frac{\partial N_i^{(4Q)}}{\partial \xi}\cdot y_i^{(e)}\right) \\ \sum_{i=1}^{4}\left(\frac{\partial N_i^{(4Q)}}{\partial \eta}\cdot x_i^{(e)}\right) & \sum_{i=1}^{4}\left(\frac{\partial N_i^{(4Q)}}{\partial \eta}\cdot y_i^{(e)}\right) \end{bmatrix} = \left[N_{,\xi}^{(4Q)}\right]\left[\left\{x^{(e)}\right\} \ \left\{y^{(e)}\right\}\right] \tag{14.7.6}$$

where

$$\left[N_{,\xi}^{(4Q)}\right] = \begin{bmatrix} \frac{\partial N_1^{(4Q)}}{\partial \xi} & \frac{\partial N_2^{(4Q)}}{\partial \xi} & \frac{\partial N_3^{(4Q)}}{\partial \xi} & \frac{\partial N_4^{(4Q)}}{\partial \xi} \\ \frac{\partial N_1^{(4Q)}}{\partial \eta} & \frac{\partial N_2^{(4Q)}}{\partial \eta} & \frac{\partial N_3^{(4Q)}}{\partial \eta} & \frac{\partial N_4^{(4Q)}}{\partial \eta} \end{bmatrix} \tag{14.7.7}$$

Given the above, we can establish the components of the $[B^{(e)}]$ array as a special case of Equation (14.6.10) with $n = 4$. Since this array contains derivatives of shape functions with respect to the physical coordinates, we have to use the chain rule of differentiation, and of course the components of the $\left[\tilde{J}\right]$ array (i.e., the inverse of $[J]$). The key is to use the following expression, previously provided as Equation (8.5.33), to find the partial derivatives of the shape functions with respect to the spatial (Cartesian) coordinates.

$$\left[N_{,x}^{(4Q)}\right] = \begin{bmatrix} \frac{\partial N_1^{(4Q)}}{\partial x} & \frac{\partial N_2^{(4Q)}}{\partial x} & \frac{\partial N_3^{(4Q)}}{\partial x} & \frac{\partial N_4^{(4Q)}}{\partial x} \\ \frac{\partial N_1^{(4Q)}}{\partial y} & \frac{\partial N_2^{(4Q)}}{\partial y} & \frac{\partial N_3^{(4Q)}}{\partial y} & \frac{\partial N_4^{(4Q)}}{\partial y} \end{bmatrix} = \left[\tilde{J}\right]\left[N_{,\xi}^{(4Q)}\right] \tag{8.5.33}$$

Finally, we can employ Gaussian quadrature for the calculation of the various integrals (2×2 quadrature is required for full integration). The procedure to obtain the element stiffness matrix, $[k^{(e)}]$, and the part of the equivalent nodal force vector, $\left\{f_\Omega^{(e)}\right\}$, due to distributed surface forces, for the case where we use N_g quadrature points, is summarized in Box 14.7.1.

Box 14.7.1 Calculation of $[k^{(e)}]$, $\{f_{\Omega}^{(e)}\}$ for 4Q flat shell element

FOR EACH QUADRATURE POINT g ($g = 1, 2, \ldots, N_g$):

i) Establish parametric coordinates, ξ_g, η_g.

ii) Calculate the values of the shape functions at the specific point:

$$N_{1g}^{(4Q)} = N_1^{(4Q)}(\xi_g, \eta_g) = \frac{1}{4}(1-\xi_g)(1-\eta_g), \quad N_{2g}^{(4Q)} = N_2^{(4Q)}(\xi_g, \eta_g) = \frac{1}{4}(1+\xi_g)(1-\eta_g)$$

$$N_{3g}^{(4Q)} = N_3^{(4Q)}(\xi_g, \eta_g) = \frac{1}{4}(1+\xi_g)(1+\eta_g), \quad N_{4g}^{(4Q)} = N_4^{(4Q)}(\xi_g, \eta_g) = \frac{1}{4}(1-\xi_g)(1+\eta_g)$$

Also, establish the shape function array $\left[N^{(4Q)}\right]_g = \left[N^{(4Q)}(\xi_g, \eta_g)\right]$ per Equation (14.7.4c).

iii) Find the values of the block shape function array $\left[N_I^{(e)}\right]_g$ of each node $I = 1,2,3,4$ using Equation (14.6.3). Then, find the shell element shape function array:

$$\left[N^{(e)}\right]_g = \left[\left[N_1^{(e)}\right]_g \quad \left[N_2^{(e)}\right]_g \quad \left[N_3^{(e)}\right]_g \quad \left[N_4^{(e)}\right]_g\right]$$

iv) Calculate values of coordinates in physical space, $x_g = x(\xi_g, \eta_g), y_g = y(\xi_g, \eta_g)$:

$$x_g = \left[N^{(4Q)}\right]_g \left\{x^{(e)}\right\}, \quad y_g = \left[N^{(4Q)}\right]_g \left\{y^{(e)}\right\}$$

v) Find the matrix $\left[N_{,\xi}^{(4Q)}\right]_g$ using Equation (14.7.7) for $\xi = \xi_g$, $\eta = \eta_g$.

vi) Calculate the Jacobian matrix of the mapping at that point using Equation (14.7.6), as well as the Jacobian determinant, $J_g = det([J_g])$, and the inverse of the Jacobian matrix, $\left[\tilde{J}_g\right] = \left[J_g\right]^{-1}$.

vii) Calculate the derivatives of the shape functions with respect to the physical coordinates, x and y, at $\xi = \xi_g$, $\eta = \eta_g$, using Equation (8.5.33): $\left[N_{,x}^{(4Q)}\right]_g = \left[\tilde{J}_g\right]\left[N_{,\xi}^{(4Q)}\right]_g$.

viii) Find the block strain-displacement arrays, $\left[B_{mI}^{(e)}\right]_g$, $\left[B_{fI}^{(e)}\right]_g$, $\left[B_{sI}^{(e)}\right]_g$, for each node $I = 1,2,3,4$, using Equations (14.6.10a–c); then, define the shell element strain-displacement array, $[B^{(e)}]_g$, at the location of Gauss point g, using Equation (14.6.10d) for $n = 4$.

ix) Calculate $\left[\hat{D}_g\right] = \left[\hat{D}(x_g, y_g)\right]$.

x) Calculate $\left\{p_g\right\} = \left\{p(x_g, y_g\right\}$.

Finally, combine the contributions of all the Gauss points to calculate the element stiffness matrix, $[k^{(e)}]$, and distributed-force contribution to the equivalent nodal force vector, $\left\{f_{\Omega}^{(e)}\right\}$:

$$\left[k^{(e)}\right] = \int_{-1}^{1}\int_{-1}^{1}\left(\left[B^{(e)}\right]^T\left[D^{(e)}\right]\left[B^{(e)}\right] J d\xi d\eta\right) \approx \sum_{g=1}^{Ng}\left(\left[B^{(e)}\right]_g^T\left[D_g\right]\left[B^{(e)}\right]_g J_g W_g\right)$$

$$\left\{f_{\Omega}^{(e)}\right\} = \iint_{\Omega^{(e)}}\left[N^{(e)}\right]^T\{p\}dV = \int_{-1}^{1}\int_{-1}^{1}\left[N^{(e)}(\xi,\eta)\right]^T\{p(\xi,\eta)\} J d\xi d\eta \approx \sum_{g=1}^{Ng}\left(\left[N^{(e)}\right]_g^T\{p_g\} J_g W_g\right)$$

After we have obtained $[k^{(e)}]$ and $\left\{f_{\Omega}^{(e)}\right\}$, we complete the element computation, by calculating the part $\left\{f_{\Gamma}^{(e)}\right\}$ of the equivalent nodal forces due to natural boundary conditions. To this end, we use a single-variable parameterization of each boundary segment, and calculate (with one-dimensional Gaussian quadrature) the contribution of the natural boundary forces and moments on $\left\{f_{\Gamma}^{(e)}\right\}$, as summarized in Box 14.7.2.

Box 14.7.2 Calculation of $\{f_{\Gamma t}^{(e)}\}$ for 4Q Flat Shell Element

First, define a vector $\{f_\Gamma\} = \left[f_x \;\; f_y \;\; f_z \;\; m_{1\Gamma} \;\; m_{2\Gamma}\right]^T$, containing the prescribed boundary forces and/or moments (note that, if only several of the components of $\{f_\Gamma\}$ are prescribed, we can use the values of these components and simply set the value of the other boundary force/moment components equal to zero).

- If the natural boundary segment corresponds to a constant value of η, $\eta = \bar{\eta}$ (where $\bar{\eta} = -1$ or 1), we need to evaluate a one-dimensional integral with respect to ξ:

$$\left\{f_{\Gamma_t}^{(e)}\right\} = \int_{\Gamma_t^{(e)}} \left[N^{(4Q)}\right]^T \{f_\Gamma\} ds = \int_{-1}^{1} \left(\left[N^{(4Q)}(\xi,\bar{\eta})\right]^T \{f_\Gamma\} \frac{\ell_\Gamma}{2} d\xi\right)$$

$$\approx \sum_{g=1}^{Ng} \left(\left[N^{(4Q)}(\xi_g,\bar{\eta})\right]^T \{f_\Gamma\} \frac{\ell_\Gamma}{2} W_g\right)$$

where ℓ_Γ is the length of the natural boundary segment.

- If the natural boundary segment corresponds to a constant value of ξ, $\xi = \bar{\xi}$ (where $\bar{\xi} = -1$ or 1), we need to evaluate a one-dimensional integral with respect to η:

$$\left\{f_{\Gamma_t}^{(e)}\right\} = \int_{\Gamma_t^{(e)}} \left[N^{(4Q)}\right]^T \{f_\Gamma\} ds = \int_{-1}^{1} \left(\left[N^{(4Q)}(\bar{\xi},\eta)\right]^T \{f_\Gamma\} \frac{\ell_\Gamma}{2} d\eta\right)$$

$$\approx \sum_{g=1}^{Ng} \left(\left[N^{(4Q)}(\bar{\xi},\eta_g)\right]^T \{f_\Gamma\} \frac{\ell_\Gamma}{2} W_g\right)$$

Remark 14.7.1: One can verify that a shell element using the formulation presented in this section does not develop any stiffness for the rotational degrees of freedom along the local z-axis, called the *drilling degrees of freedom*! If these degrees of freedom are not connected to other elements in a three-dimensional mesh, which develop stiffness, then there may be problems in the solution of the global stiffness equations. Several studies have been focused on the definition of mathematically meaningful drilling stiffness terms for elastic shells (e.g., Hughes and Brezzi (1989)). The incorporation of such drilling stiffness is possible in many commercial finite element programs; however, it will not be discussed in this text. It is worth mentioning that if all the shell elements in a mesh are coplanar (i.e., they are all planar and lie on the same plane xy), then one can simply restrain all the drilling rotational nodal degrees-of-freedom to a zero value. ∎

14.8 Coordinate Transformations for Shell Elements

The considerations of the previous Sections were based on the assumption that the shell x-and y-axes coincide with the global x- and y- coordinate axes. In general analyses, the various flat shell elements may not all lie on the same plane, as shown, e.g., in Figure 14.12 for the analysis of a wall structure with shell elements. Obviously, the normal direction (local z-axis) is not identical for the various elements in this case.

The expressions established in Section 14.7 for $[k^{(e)}]$, $\{f^{(e)}\}$, etc. are always valid for the element *local coordinate system*, in which axes x and y lie in the plane of the element and axis z is normal to the plane of the element's mid-surface.

For cases involving elements that do not all have the same normal direction, the global equations must be formulated in a unique, *global (or structural) coordinate system*, XYZ. The stiffness matrix and equivalent nodal force/moment vector of each element are initially established in the local coordinate system. Just like for beams in Section 13.7 (and for two-dimensional trusses in Section B.8), it is necessary to establish a *coordinate transformation equation*, from the local to the global coordinate system. We must apply the coordinate transformation equations to the element stiffness matrix and equivalent nodal force vector before we conduct the assembly operation. In other words, the assembly equations are conducted for the stiffness and nodal force expressed in the global coordinate system. This section is dedicated to the formulation of the coordinate transformation for shells. As a starting point, we will assume that we are given the nodal coordinates in the global coordinate system. Specifically, for each node I, we will have a triad of coordinate values X_I, Y_I, Z_I.

The first step toward establishing the local coordinate system of a quadrilateral shell element is to calculate the unit vectors in the directions of the diagonals, as shown in Figure 14.13. The figure also provides the mathematical

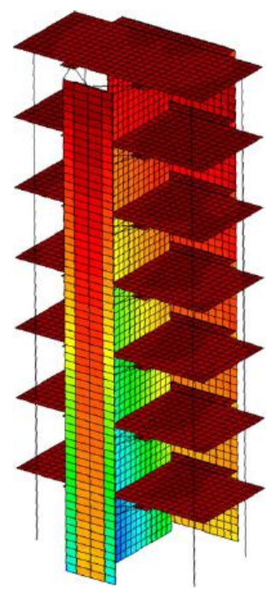
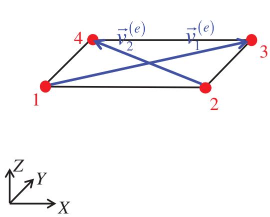


Figure 14.12 Flat shell element model for a reinforced concrete wall.



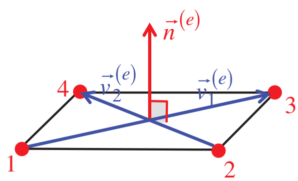
$$\vec{v}_1^{(e)} = \begin{Bmatrix} v_{1x}^{(e)} \\ v_{1y}^{(e)} \\ v_{1z}^{(e)} \end{Bmatrix} = \frac{1}{\ell_{31}^{(e)}} \begin{Bmatrix} X_3^{(e)} - X_1^{(e)} \\ Y_3^{(e)} - Y_1^{(e)} \\ Z_3^{(e)} - Z_1^{(e)} \end{Bmatrix}, \quad \ell_{31}^{(e)} = \sqrt{\left(X_3^{(e)} - X_1^{(e)}\right)^2 + \left(Y_3^{(e)} - Y_1^{(e)}\right)^2 + \left(Z_3^{(e)} - Z_1^{(e)}\right)^2}$$

$$\vec{v}_2^{(e)} = \begin{Bmatrix} v_{2x}^{(e)} \\ v_{2y}^{(e)} \\ v_{2z}^{(e)} \end{Bmatrix} = \frac{1}{\ell_{41}^{(e)}} \begin{Bmatrix} X_4^{(e)} - X_2^{(e)} \\ Y_4^{(e)} - Y_2^{(e)} \\ Z_4^{(e)} - Z_2^{(e)} \end{Bmatrix}, \quad \ell_{41}^{(e)} = \sqrt{\left(X_4^{(e)} - X_2^{(e)}\right)^2 + \left(Y_4^{(e)} - Y_2^{(e)}\right)^2 + \left(Z_4^{(e)} - Z_2^{(e)}\right)^2}$$

Figure 14.13 Establishing unit vectors in the directions of the diagonals of a quadrilateral shell element.

equations giving the unit vector $\vec{v}_1^{(e)}$ in the direction of diagonal (1-3), and the unit vector $\vec{v}_2^{(e)}$ in the direction of diagonal (2-4).

Having obtained the two unit vectors, $\vec{v}_1^{(e)}$ and $\vec{v}_2^{(e)}$, we can use the fact that these vectors lie in the plane of the shell to find the unit vector $\vec{n}^{(e)}$ normal to the plane of the element. This vector is obtained as the cross product (or exterior product) of the two "diagonal" vectors[2]. The normal vector $\vec{n}^{(e)}$ and the mathematical expression yielding this vector are provided in Figure 14.14.



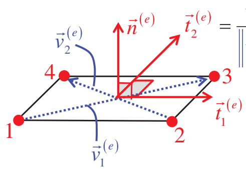
$$\vec{n}^{(e)} = \frac{\vec{v}_1^{(e)} \times \vec{v}_2^{(e)}}{\left\| \vec{v}_1^{(e)} \times \vec{v}_2^{(e)} \right\|}, \quad \vec{v}_1^{(e)} \times \vec{v}_2^{(e)} = \begin{Bmatrix} v_{1y}^{(e)} \cdot v_{2z}^{(e)} - v_{1z}^{(e)} \cdot v_{2y}^{(e)} \\ v_{1z}^{(e)} \cdot v_{2x}^{(e)} - v_{1x}^{(e)} \cdot v_{2z}^{(e)} \\ v_{1x}^{(e)} \cdot v_{2y}^{(e)} - v_{1y}^{(e)} \cdot v_{2x}^{(e)} \end{Bmatrix}$$

Figure 14.14 Establishing the unit normal vector for a quadrilateral shell element.

Given $\vec{v}_1^{(e)}$ and $\vec{v}_2^{(e)}$, we also establish a pair of unit vectors, $\vec{t}_1^{(e)}$ and $\vec{t}_2^{(e)}$, providing the local x- and local y-axes, respectively, of the element. The local z-direction of the element is given by vector $\vec{n}^{(e)}$. The orientation of vectors $\vec{t}_1^{(e)}$ and $\vec{t}_2^{(e)}$ and the mathematical expressions giving these two vectors are shown in Figure 14.15.

After the establishment of the local coordinate axes, we can write the coordinate transformation equations for nodal displacements and rotations. The equations for the nodal displacements are obtained as a special case of Equation (7.6.2). An important consideration pertaining to the nodal rotations will be used here: *for small nodal rotations (which is always an assumption made in linear analysis!), the three components of nodal rotation can be considered as the three components of a vector*. This means that the same coordinate transformation equations can be used for nodal displacements and nodal rotations.

Let us start with the nodal displacements. If, for a nodal point I of an element, we have the nodal displacements $\left[u_{Ix}^{(e)} \; u_{Iy}^{(e)} \; u_{Iz}^{(e)}\right]^T$ in the global coordinate system, then the nodal displacements of the same node in the local coordinate system, $\left[u_{Ix}^{\prime(e)} \; u_{Iy}^{\prime(e)} \; u_{Iz}^{\prime(e)}\right]^T$, are given by the following equation.



$$\vec{t}_2^{(e)} = \frac{\vec{v}_1^{(e)} + \vec{v}_2^{(e)}}{\left\| \vec{v}_1^{(e)} + \vec{v}_2^{(e)} \right\|}, \quad \vec{t}_1^{(e)} = \frac{\vec{v}_1^{(e)} - \vec{v}_2^{(e)}}{\left\| \vec{v}_1^{(e)} - \vec{v}_2^{(e)} \right\|}$$

$$Note: \; \vec{n}^{(e)} = \frac{\vec{v}_1^{(e)} \times \vec{v}_2^{(e)}}{\left\| \vec{v}_1^{(e)} \times \vec{v}_2^{(e)} \right\|} = \frac{\vec{t}_1^{(e)} \times \vec{t}_2^{(e)}}{\left\| \vec{t}_1^{(e)} \times \vec{t}_2^{(e)} \right\|}$$

Figure 14.15 Establishing the unit vectors giving the local in-plane coordinate axes of a quadrilateral planar shell element.

2 It is known from vector calculus that the cross-product of two vectors is a third vector, perpendicular to the plane defined by the two vectors.

$$\begin{Bmatrix} u_{Ix}^{\prime(e)} \\ u_{Iy}^{\prime(e)} \\ u_{Iz}^{\prime(e)} \end{Bmatrix} = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} \begin{Bmatrix} u_{Ix}^{(e)} \\ u_{Iy}^{(e)} \\ u_{Iz}^{(e)} \end{Bmatrix} \tag{14.8.1}$$

The first row of the coefficient array in Equation (14.8.1) consists of the three components of the unit vector $\vec{t}_1^{(e)}$, the second row of the array includes the three components of vector $\vec{t}_2^{(e)}$, and the third row includes the components of vector $\vec{n}^{(e)}$.

Similarly, if we have the nodal rotations $\left[\theta_{Ix}^{(e)} \;\; \theta_{Iy}^{(e)} \;\; \theta_{Iz}^{(e)}\right]^T$ for a nodal point I in the global coordinate system, then the corresponding rotations in the local coordinate system, $\left[\theta_{Ix}^{\prime(e)} \;\; \theta_{Iy}^{\prime(e)} \;\; \theta_{Iz}^{\prime(e)}\right]^T$, are given by the following expression.

$$\begin{Bmatrix} \theta_{Ix}^{\prime(e)} \\ \theta_{Iy}^{\prime(e)} \\ \theta_{Iz}^{\prime(e)} \end{Bmatrix} = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} \begin{Bmatrix} \theta_{Ix}^{(e)} \\ \theta_{Iy}^{(e)} \\ \theta_{Iz}^{(e)} \end{Bmatrix} \tag{14.8.2}$$

We can collectively cast Equations (14.8.1) and (14.8.2) into a single matrix expression:

$$\left\{U_I^{\prime(e)}\right\} = \begin{Bmatrix} u_{Ix}^{\prime(e)} \\ u_{Iy}^{\prime(e)} \\ u_{Iz}^{\prime(e)} \\ \theta_{Ix}^{\prime(e)} \\ \theta_{Iy}^{\prime(e)} \\ \theta_{Iz}^{\prime(e)} \end{Bmatrix} = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} & 0 & 0 & 0 \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} & 0 & 0 & 0 \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ 0 & 0 & 0 & t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ 0 & 0 & 0 & n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} \begin{Bmatrix} u_{Ix}^{(e)} \\ u_{Iy}^{(e)} \\ u_{Iz}^{(e)} \\ \theta_{Ix}^{(e)} \\ \theta_{Iy}^{(e)} \\ \theta_{Iz}^{(e)} \end{Bmatrix} = \left[r^{(e)}\right]\left\{U_I^{(e)}\right\} \tag{14.8.3}$$

where $[r^{(e)}]$ is the coordinate transformation array for the six degrees of freedom of each node in element e, given by:

$$\left[r^{(e)}\right] = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} & 0 & 0 & 0 \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} & 0 & 0 & 0 \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ 0 & 0 & 0 & t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ 0 & 0 & 0 & n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} \tag{14.8.4}$$

We can establish a block-matrix coordinate transformation equation for the full nodal displacement/rotation vector of a four-node flat shell element e (containing the nodal displacements and rotations of all four nodal points):

$$\begin{Bmatrix} \{U_1^{\prime(e)}\} \\ \{U_2^{\prime(e)}\} \\ \{U_3^{\prime(e)}\} \\ \{U_4^{\prime(e)}\} \end{Bmatrix} = \begin{bmatrix} [r^{(e)}] & [0] & [0] & [0] \\ [0] & [r^{(e)}] & [0] & [0] \\ [0] & [0] & [r^{(e)}] & [0] \\ [0] & [0] & [0] & [r^{(e)}] \end{bmatrix} \begin{Bmatrix} \{U_1^{(e)}\} \\ \{U_2^{(e)}\} \\ \{U_3^{(e)}\} \\ \{U_4^{(e)}\} \end{Bmatrix}$$

$$\rightarrow \{U^{\prime(e)}\} = [R^{(e)}]\{U^{(e)}\} \tag{14.8.5}$$

where $[R^{(e)}]$ is the (24×24) coordinate transformation array for a four-node flat shell element:

$$[R^{(e)}] = \begin{bmatrix} [r^{(e)}] & [0] & [0] & [0] \\ [0] & [r^{(e)}] & [0] & [0] \\ [0] & [0] & [r^{(e)}] & [0] \\ [0] & [0] & [0] & [r^{(e)}] \end{bmatrix} \tag{14.8.6}$$

It can easily be proven that, for nodal forces and moments, the following transformation equation applies:

$$\{F^{(e)}\} = [R^{(e)}]^T \{F^{\prime(e)}\} \tag{14.8.7}$$

Finally, we can obtain (with a procedure similar to that employed in Section 13.7) the coordinate transformation equation for the element stiffness matrix:

$$[k^{(e)}] = [R^{(e)}]^T [k^{\prime(e)}] [R^{(e)}] \tag{14.8.8}$$

where $[k^{\prime(e)}]$ is the stiffness matrix in the local coordinate system, and $[k^{(e)}]$ is the same quantity in the global coordinate system. In summary, if we need to conduct coordinate transformation for a shell element, we use Equations (14.6.13) and (14.6.21) to find $[k^{\prime(e)}]$ and $\{f^{\prime(e)}\}$, respectively. Then, we find $[k^{(e)}]$ and $\{f^{(e)}\}$ (the stiffness matrix and equivalent nodal force vector in the global coordinate system) using Equations (14.8.8) and (14.8.7), respectively. Finally, we can conduct the assembly operations (Equations 14.6.24 and 14.6.25) with the arrays $[k^{(e)}]$ and vectors $\{f^{(e)}\}$ of all elements.

Remark 14.8.1: It is important to remember that the physical coordinates of the nodal points must be expressed in the local coordinate system, before being used in the calculations described in Section 14.7. If we have the coordinates $\begin{bmatrix} X_I^{(e)} & Y_I^{(e)} & Z_I^{(e)} \end{bmatrix}^T$ of nodal point I of element e in the global coordinate system, then the corresponding coordinates of the same nodal point in the global coordinate system can be obtained from the following equation.

$$\begin{Bmatrix} x_I'^{(e)} \\ y_I'^{(e)} \\ z_I'^{(e)} \end{Bmatrix} = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} \begin{Bmatrix} X_I^{(e)} \\ Y_I^{(e)} \\ Z_I^{(e)} \end{Bmatrix} \tag{14.8.9}$$

■

Remark 14.8.2: For a flat shell element e, the coordinate $z'^{(e)}$ (corresponding to the coordinate along the local axis $\vec{n}^{(e)}$) has the same value for all nodal points in the element. ■

Example 14.1: Determination of $[k^{(e)}]$ and $\{f^{(e)}\}$ for a four-node quadrilateral shell element

In this example, we will find the stiffness matrix and equivalent nodal force/moment vector for the four-node element shown in Figure 14.16. The coordinates (in the global coordinate system) of the four nodal points are also provided in the same figure. The material is isotropic, linearly elastic, with $E = 5000$ ksi, $\nu = 0.2$, and the thickness of the element is $d = 6$ inches. The element is subjected to a distributed force (per unit surface area of the shell) of 1kip/in.^{2}, along the negative global axis Z. No other loadings are applied.

We begin by determining the 3 × 3 coordinate transformation block matrix, $[r^{(e)}]$:

$$\ell_{31}^{(e)} = \sqrt{\left(X_3^{(e)} - X_1^{(e)}\right)^2 + \left(Y_3^{(e)} - Y_1^{(e)}\right)^2 + \left(Z_3^{(e)} - Z_1^{(e)}\right)^2}$$

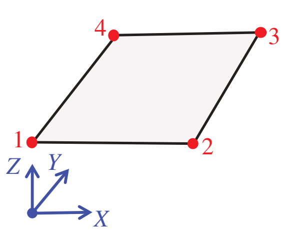
$$= \sqrt{(5-0)^2 + (4-0)^2 + (2-1)^2} = 6.481\,\text{in.}$$

$$\ell_{41}^{(e)} = \sqrt{\left(X_4^{(e)} - X_2^{(e)}\right)^2 + \left(Y_4^{(e)} - Y_2^{(e)}\right)^2 + \left(Z_4^{(e)} - Z_2^{(e)}\right)^2}$$

$$= \sqrt{(1-5)^2 + (4-0)^2 + (2-1)^2} = 5.745\,\text{in.}$$

We can now find the unit vectors $\vec{v}_1^{(e)}, \vec{v}_2^{(e)}$:

$$\vec{v}_1^{(e)} = \begin{Bmatrix} v_{1x}^{(e)} \\ v_{1y}^{(e)} \\ v_{1z}^{(e)} \end{Bmatrix} = \frac{1}{\ell_{31}^{(e)}} \begin{Bmatrix} X_3^{(e)} - X_1^{(e)} \\ Y_3^{(e)} - Y_1^{(e)} \\ Z_3^{(e)} - Z_1^{(e)} \end{Bmatrix} = \frac{1}{6.481} \begin{Bmatrix} 5-0 \\ 4-0 \\ 2-1 \end{Bmatrix} = \begin{Bmatrix} 0.7715 \\ 0.6172 \\ 0.1543 \end{Bmatrix}$$



Node	X (in.)	Y (in.)	Z (in.)
1	0	0	1
2	5	0	1
3	5	4	2
4	1	4	2

Figure 14.16 Example quadrilateral shell element.

$$\vec{v}_2^{(e)} = \begin{Bmatrix} v_{2x}^{(e)} \\ v_{2y}^{(e)} \\ v_{2z}^{(e)} \end{Bmatrix} = \frac{1}{\ell_{41}^{(e)}} \begin{Bmatrix} X_4^{(e)} - X_2^{(e)} \\ Y_4^{(e)} - Y_2^{(e)} \\ Z_4^{(e)} - Z_2^{(e)} \end{Bmatrix} = \frac{1}{5.745} \begin{Bmatrix} 1-5 \\ 4-0 \\ 2-1 \end{Bmatrix} = \begin{Bmatrix} -0.6963 \\ 0.6963 \\ 0.1741 \end{Bmatrix}$$

We can also find the local coordinate axis

$$\vec{n}^{(e)} = \frac{\vec{v}_1^{(e)} \times \vec{v}_2^{(e)}}{\left\| \vec{v}_1^{(e)} \times \vec{v}_2^{(e)} \right\|} = \begin{Bmatrix} 0 \\ -0.2425 \\ 0.9701 \end{Bmatrix}$$

Finally, we can establish the other two, in-plane, local coordinate axes of the element, $\vec{t}_1^{(e)}$ and $\vec{t}_2^{(e)}$:

$$\vec{t}_1^{(e)} = \frac{\vec{v}_1^{(e)} - \vec{v}_2^{(e)}}{\left\| \vec{v}_1^{(e)} - \vec{v}_2^{(e)} \right\|} = \begin{Bmatrix} 0.9985 \\ -0.0538 \\ -0.0135 \end{Bmatrix}, \vec{t}_2^{(e)} = \frac{\vec{v}_1^{(e)} + \vec{v}_2^{(e)}}{\left\| \vec{v}_1^{(e)} + \vec{v}_2^{(e)} \right\|} = \begin{Bmatrix} 0.0555 \\ -0.9686 \\ 0.2422 \end{Bmatrix}$$

We now establish the 3×3 array:

$$\begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} = \begin{bmatrix} 0.9985 & -0.0538 & -0.0135 \\ 0.0555 & 0.9686 & 0.2422 \\ 0 & -0.2425 & 0.9701 \end{bmatrix}$$

The coordinates of each node I in the local coordinate system of the element can be obtained from:

$$\begin{Bmatrix} x_I'^{(e)} \\ y_I'^{(e)} \\ z_I'^{(e)} \end{Bmatrix} = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} \begin{Bmatrix} X_I^{(e)} \\ Y_I^{(e)} \\ Z_I^{(e)} \end{Bmatrix}.$$

By substitution, we have:

$$\begin{Bmatrix} x_1'^{(e)} \\ y_1'^{(e)} \\ z_1'^{(e)} \end{Bmatrix} = \begin{Bmatrix} -0.0135 \\ 0.2422 \\ 0.9701 \end{Bmatrix}, \begin{Bmatrix} x_2'^{(e)} \\ y_2'^{(e)} \\ z_2'^{(e)} \end{Bmatrix} = \begin{Bmatrix} 4.9789 \\ 0.5195 \\ 0.9701 \end{Bmatrix}, \begin{Bmatrix} x_3'^{(e)} \\ y_3'^{(e)} \\ z_3'^{(e)} \end{Bmatrix} = \begin{Bmatrix} 4.7502 \\ 4.6362 \\ 0.9701 \end{Bmatrix}, \begin{Bmatrix} x_4'^{(e)} \\ y_4'^{(e)} \\ z_4'^{(e)} \end{Bmatrix} = \begin{Bmatrix} 0.7563 \\ 4.4144 \\ 0.9701 \end{Bmatrix}$$

It is worth noticing that the value of local z-coordinate is identical for the four-nodal points (this must always be the case for a planar shell element). We can now treat the element as an isoparametric 4Q one, using the values of local x- and y-coordinate for the four nodal points.

The given distributed force vector is expressed in the global coordinate system (because we know that one of the components, along the global Z axis, is nonzero). When we calculate the equivalent nodal force vector, we must have the three components of distributed force (per unit area of the shell) along the local axes of the element (so that the first two components are membrane forces, the third component is a force perpendicular to the plane of the element). The same coordinate transformation equation that applies for nodal displacements and coordinates must be satisfied for the distributed forces:

$$\begin{Bmatrix} p_x'^{(e)} \\ p_y'^{(e)} \\ p_z'^{(e)} \end{Bmatrix} = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix} \begin{Bmatrix} p_x^{(e)} \\ p_y^{(e)} \\ p_z^{(e)} \end{Bmatrix}$$

$$= \begin{bmatrix} 0.9985 & -0.0538 & -0.0135 \\ 0.0555 & 0.9686 & 0.2422 \\ 0 & -0.2425 & 0.9701 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -1 \end{Bmatrix} = \begin{Bmatrix} 0.0135 \\ -0.2422 \\ -0.9701 \end{Bmatrix}$$

The five-component distributed force/moment vector in the local coordinate system for the specific element can be written as: $\left\{p'^{(e)}\right\} = \left[p_x'^{(e)} \quad p_y'^{(e)} \quad p_z'^{(e)} \quad 0 \quad 0\right]^T$, where we have accounted for the fact that, in the specific example, we do not have any distributed moments in our element.

We can also establish the block-matrices for the generalized stress-strain matrix of the element, which is constant:

$$\left[\hat{D}_m\right] = \frac{E \cdot d}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 31250 & 6250 & 0 \\ 6250 & 31250 & 0 \\ 0 & 0 & 12500 \end{bmatrix},$$

$$\left[\hat{D}_f\right] = \frac{E \cdot d^3}{12(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 93750 & 18750 & 0 \\ 18750 & 93750 & 0 \\ 0 & 0 & 37500 \end{bmatrix}$$

$$\left[\hat{D}_s\right] = \kappa \cdot G \cdot d \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 10417 & 0 \\ 0 & 10417 \end{bmatrix}$$

The generalized stress-strain matrix of the element can be written per Equation (14.4.42):

$$[\hat{D}] = \begin{bmatrix} [\hat{D}_m] & [0] & [0] \\ [0] & [\hat{D}_f] & [0] \\ [0] & [0] & [\hat{D}_s] \end{bmatrix}$$

$$= \begin{bmatrix} \begin{bmatrix} 31250 & 6250 & 0 \\ 6250 & 31250 & 0 \\ 0 & 0 & 12500 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 93750 & 18750 & 0 \\ 18750 & 93750 & 0 \\ 0 & 0 & 37500 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 10417 & 0 \\ 0 & 10417 \end{bmatrix} \end{bmatrix}$$

We will conduct in detail the computations for the first integration (Gauss) point of the element, with parametric coordinates $\xi_1 = \frac{1}{\sqrt{3}}$, $\eta_1 = -\frac{1}{\sqrt{3}}$

$$N_1^{(4Q)} = N_1^{(4Q)}(\xi = \xi_1, \eta = \eta_1) = \frac{1}{4}(1-\xi_1)(1-\eta_1) = 0.6220$$

$$N_2^{(4Q)} = N_2^{(4Q)}(\xi = \xi_1, \eta = \eta_1) = \frac{1}{4}(1+\xi_1)(1-\eta_1) = 0.1667$$

$$N_3^{(4Q)} = N_3^{(4Q)}(\xi = \xi_1, \eta = \eta_1) = \frac{1}{4}(1+\xi_1)(1+\eta_1) = 0.0447$$

$$N_4^{(4Q)} = N_4^{(4Q)}(\xi = \xi_1, \eta = \eta_1) = \frac{1}{4}(1-\xi_1)(1+\eta_1) = 0.1667$$

$$\left[N_{,\xi}^{(4Q)}\right]_1 = \begin{bmatrix} \frac{\partial N_1^{(4Q)}}{\partial \xi} & \frac{\partial N_2^{(4Q)}}{\partial \xi} & \frac{\partial N_3^{(4Q)}}{\partial \xi} & \frac{\partial N_4^{(4Q)}}{\partial \xi} \\ \frac{\partial N_1^{(4Q)}}{\partial \eta} & \frac{\partial N_2^{(4Q)}}{\partial \eta} & \frac{\partial N_3^{(4Q)}}{\partial \eta} & \frac{\partial N_4^{(4Q)}}{\partial \eta} \end{bmatrix}_{\substack{\xi=\xi_1 \\ \eta=\eta_1}}$$

$$= \begin{bmatrix} -0.3943 & 0.3943 & 0.1057 & -0.1057 \\ -0.3943 & -0.1057 & 0.1057 & 0.3943 \end{bmatrix}$$

$$[J]_1 = \left[N_{,\xi}^{(4Q)}\right]_1 \left[\left\{x'^{(e)}\right\} \ \left\{y'^{(e)}\right\}\right]$$

$$= \begin{bmatrix} -0.3943 & 0.3943 & 0.1057 & -0.1057 \\ -0.3943 & -0.1057 & 0.1057 & 0.3943 \end{bmatrix} \left[\begin{Bmatrix} -0.0135 \\ 4.9789 \\ 4.7502 \\ 0.7563 \end{Bmatrix} \begin{Bmatrix} 0.2422 \\ 0.5195 \\ 4.6362 \\ 4.4144 \end{Bmatrix}\right]$$

$$= \begin{bmatrix} 2.9307 & 0.1328 \\ 0.2794 & 2.0802 \end{bmatrix}$$

$$J_1 = \det\left([J]_1\right) = 4.9361$$

$$\left[\tilde{J}\right]_1 = \left([J]_1\right)^{-1} = \begin{bmatrix} 0.4214 & -0.0269 \\ -0.0566 & 0.4843 \end{bmatrix}$$

$$\left[N_{,x}^{(4Q)}\right] = \begin{bmatrix} \dfrac{\partial N_1^{(4Q)}}{\partial x} & \dfrac{\partial N_2^{(4Q)}}{\partial x} & \dfrac{\partial N_3^{(4Q)}}{\partial x} & \dfrac{\partial N_4^{(4Q)}}{\partial x} \\ \dfrac{\partial N_1^{(4Q)}}{\partial y} & \dfrac{\partial N_2^{(4Q)}}{\partial y} & \dfrac{\partial N_3^{(4Q)}}{\partial y} & \dfrac{\partial N_4^{(4Q)}}{\partial y}. \end{bmatrix}_1 = \left[\tilde{J}\right]_1 \left[N_{,\xi}^{(4Q)}\right]_1$$

$$= \begin{bmatrix} -0.1556 & 0.1690 & 0.0417 & -0.0551 \\ -0.1687 & -0.0735 & 0.0452 & 0.1970 \end{bmatrix}$$

We can establish the 5×6 block shape-function arrays for the four nodal points, evaluated at Gauss point 1. For each node I, we have:

$$\left[N_I^{(e)}\right]_1 = \begin{bmatrix} N_I^{(4Q)}(\xi,\eta) & 0 & 0 & 0 & 0 & 0 \\ 0 & N_I^{(4Q)}(\xi,\eta) & 0 & 0 & 0 & 0 \\ 0 & 0 & N_I^{(4Q)}(\xi,\eta) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -N_I^{(4Q)}(\xi,\eta) & 0 \\ 0 & 0 & 0 & N_I^{(4Q)}(\xi,\eta) & 0 & 0 \end{bmatrix}_{\substack{\xi=\xi_1 \\ \eta=\eta_1}}$$

The block shape function array at Gauss point 1 for nodes 1, 2, 3 and 4 is:

$$\left[N_1^{(e)}\right]_1 = \begin{bmatrix} 0.6220 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.6220 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6220 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.6220 & 0 \\ 0 & 0 & 0 & 0.6220 & 0 & 0 \end{bmatrix}$$

$$\left[N_2^{(e)}\right]_1 = \begin{bmatrix} 0.1667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1667 & 0 \\ 0 & 0 & 0 & 0.1667 & 0 & 0 \end{bmatrix}$$

$$\left[N_3^{(e)}\right]_1 = \begin{bmatrix} 0.0447 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0447 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0447 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0447 & 0 \\ 0 & 0 & 0 & 0.0447 & 0 & 0 \end{bmatrix}$$

$$\left[N_4^{(e)}\right]_1 = \begin{bmatrix} 0.1667 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1667 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1667 & 0 \\ 0 & 0 & 0 & 0.1667 & 0 & 0 \end{bmatrix}$$

Given the four block-shape-function arrays at Gauss point 1, we can also obtain the 5×24 shape function array $[N^{(e)}]$ of the shell element, using Equation (14.6.6) for the special case $n = 4$:

$$\left[N^{(e)}\right]_1 = \left[\left[N_1^{(e)}\right]_1 \quad \left[N_2^{(e)}\right]_1 \quad \left[N_3^{(e)}\right]_1 \quad \left[N_4^{(e)}\right]_1\right]$$

We now proceed to find the block strain-displacement arrays for the four nodal points, evaluated at Gauss point 1:

For each node I, we have:

$$\left[B_{mI}^{(e)}\right]_1 = \begin{bmatrix} \frac{\partial N_I^{(4Q)}}{\partial x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\partial N_I^{(4Q)}}{\partial y} & 0 & 0 & 0 & 0 \\ \frac{\partial N_I^{(4Q)}}{\partial y} & \frac{\partial N_I^{(4Q)}}{\partial x} & 0 & 0 & 0 & 0 \end{bmatrix}_1, \quad \left[B_{fI}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{\partial N_I^{(4Q)}}{\partial x} & 0 \\ 0 & 0 & 0 & \frac{\partial N_I^{(4Q)}}{\partial y} & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial N_I^{(4Q)}}{\partial y} & -\frac{\partial N_I^{(4Q)}}{\partial x} & 0 \end{bmatrix}_1,$$

$$\left[B_{sI}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & \frac{\partial N_I^{(4Q)}}{\partial x} & 0 & N_I^{(4Q)} & 0 \\ 0 & 0 & \frac{\partial N_I^{(4Q)}}{\partial y} & -N_I^{(4Q)} & 0 & 0 \end{bmatrix}_1$$

The block strain-displacement arrays at Gauss point 1 for node 1 are:

$$\left[B_{mI}^{(e)}\right]_1 = \begin{bmatrix} -01556 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1687 & 0 & 0 & 0 & 0 \\ -0.1687 & -0.1556 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left[B_{f1}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.1556 & 0 \\ 0 & 0 & 0 & -0.1687 & 0 & 0 \\ 0 & 0 & 0 & -0.1556 & 0.1687 & 0 \end{bmatrix}$$

$$\left[B_{s1}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & -0.1556 & 0 & 0.6220 & 0 \\ 0 & 0 & -0.1687 & -0.622 & 0 & 0 \end{bmatrix}$$

The strain-displacement arrays at Gauss point 1 for node 2 are:

$$\left[B_{m2}^{(e)}\right]_1 = \begin{bmatrix} 0.1690 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.0735 & 0 & 0 & 0 & 0 \\ -0.0735 & 0.1690 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left[B_{f2}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.1690 & 0 \\ 0 & 0 & 0 & -0.0735 & 0 & 0 \\ 0 & 0 & 0 & 0.1690 & 0.0735 & 0 \end{bmatrix}$$

$$\left[B_{s2}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & 0.1690 & 0 & 0.167 & 0 \\ 0 & 0 & -0.0735 & -0.1667 & 0 & 0 \end{bmatrix}$$

The same arrays at Gauss point 1 for node 3 are:

$$\left[B_{m3}^{(e)}\right]_1 = \begin{bmatrix} 0.0417 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0452 & 0 & 0 & 0 & 0 \\ 0.0452 & 0.0417 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left[B_{f3}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.0417 & 0 \\ 0 & 0 & 0 & 0.0452 & 0 & 0 \\ 0 & 0 & 0 & 0.0417 & -0.0452 & 0 \end{bmatrix}$$

$$\left[B_{s3}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & 0.0417 & 0 & 0.0447 & 0 \\ 0 & 0 & 0.0452 & -0.0447 & 0 & 0 \end{bmatrix}$$

The block strain-displacement arrays at Gauss point 1 for node 4 are:

$$\left[B_{m4}^{(e)}\right]_1 = \begin{bmatrix} -0.0551 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.1970 & 0 & 0 & 0 & 0 \\ 0.1970 & -0.0551 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\left[B_{f4}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.0551 & 0 \\ 0 & 0 & 0 & 0.1970 & 0 & 0 \\ 0 & 0 & 0 & -0.0551 & -0.1970 & 0 \end{bmatrix}$$

$$\left[B_{s4}^{(e)}\right]_1 = \begin{bmatrix} 0 & 0 & -0.0551 & 0 & 0.1667 & 0 \\ 0 & 0 & 0.1970 & -0.1667 & 0 & 0 \end{bmatrix}$$

We can write the 8×24 generalized strain-displacement matrix of Gauss point 1, $[B^{(e)}]_1$, in block-matrix form in accordance with Equation (14.6.10d), for the special case $n = 4$:

$$\underset{8\times 24}{\left[B^{(e)}\right]_1} = \begin{bmatrix} \begin{bmatrix} \underset{3\times 6}{\left[B_{m1}^{(e)}\right]_1} \\ \underset{3\times 6}{\left[B_{f1}^{(e)}\right]_1} \\ \underset{2\times 6}{\left[B_{s1}^{(e)}\right]_1} \end{bmatrix} & \begin{bmatrix} \underset{3\times 6}{\left[B_{m2}^{(e)}\right]_1} \\ \underset{3\times 6}{\left[B_{f2}^{(e)}\right]_1} \\ \underset{2\times 6}{\left[B_{s2}^{(e)}\right]_1} \end{bmatrix} & \begin{bmatrix} \underset{3\times 6}{\left[B_{m3}^{(e)}\right]_1} \\ \underset{3\times 6}{\left[B_{f3}^{(e)}\right]_1} \\ \underset{2\times 6}{\left[B_{s3}^{(e)}\right]_1} \end{bmatrix} & \begin{bmatrix} \underset{3\times 6}{\left[B_{m4}^{(e)}\right]_1} \\ \underset{3\times 6}{\left[B_{f4}^{(e)}\right]_1} \\ \underset{2\times 6}{\left[B_{s4}^{(e)}\right]_1} \end{bmatrix} \end{bmatrix};$$

for clarity, we have written below each matrix its corresponding dimensions.

The contributions of Gauss point 1 to the stiffness matrix in the local coordinate system, $[k'^{(e)}]$, and to the equivalent nodal force/moment vector, $\{f'^{(e)}\}$, in the local coordinate system, are given by the expressions $\left(\left[B^{(e)}\right]_1\right)^T \left[D^{(e)}\right]_1 \left[B^{(e)}\right]_1 J_1 \cdot W_1$ and $\left(\left[N^{(e)}\right]_1\right)^T \left\{p^{(e)}\right\}_1 J_1 \cdot W_1$, respectively, where $W_1 = 1$ is the weight coefficient of the first Gauss point for a 2×2 quadrature rule.

We use the same exact computations for Gauss points 2 $\left(\xi_2 = 1/\sqrt{3}, \eta_2 = -1/\sqrt{3}\right)$, 3 $\left(\xi_3 = 1/\sqrt{3}, \eta_3 = 1/\sqrt{3}\right)$, and 4 $\left(\xi_4 = -1/\sqrt{3}, \eta_4 = 1/\sqrt{3}\right)$. We can then sum the Gauss point contributions to obtain $[k'^{(e)}]$ and $\{f'^{(e)}\}$:

$$\left[k'^{(e)}\right] = \left(\left[B^{(e)}\right]_1\right)^T \left[D^{(e)}\right]_1 \left[B^{(e)}\right]_1 J_1 \cdot W_1 + \left(\left[B^{(e)}\right]_2\right)^T \left[D^{(e)}\right]_2 \left[B^{(e)}\right]_2 J_2 \cdot W_2 + \left(\left[B^{(e)}\right]_3\right)^T \left[D^{(e)}\right]_3 \left[B^{(e)}\right]_3 J_3 \cdot W_3 + \left(\left[B^{(e)}\right]_4\right)^T \left[D^{(e)}\right]_4 \left[B^{(e)}\right]_4 J_4 \cdot W_4$$

$$= \left[\begin{array}{cccccccccccccccccccccccc}
12213 & 4005 & 0 & 0 & 0 & 0 & -6641 & -1413 & 0 & 0 & 0 & 0 & -5863 & -4342 & 0 & 0 & 0 & 0 & 291 & 1750 & 0 & 0 & 0 & 0 \\
4005 & 13087 & 0 & 0 & 0 & 0 & 1712 & 2563 & 0 & 0 & 0 & 0 & -4342 & -6283 & 0 & 0 & 0 & 0 & -1375 & -9367 & 0 & 0 & 0 & 0 \\
0 & 0 & 6024 & 7331 & -6762 & 0 & 0 & 0 & -971 & 0 & 0 & 0 & 0 & 0 & -2892 & 0 & 0 & 0 & 0 & 0 & -2161 & 0 & 0 & 0 \\
0 & 0 & 7331 & 61928 & -12016 & 0 & 0 & 0 & 4803 & 7689 & -5137 & 0 & 0 & 0 & -3665 & -18848 & 13027 & 0 & 0 & 0 & -8469 & -28101 & 4126 & 0 \\
0 & 0 & -6762 & -12016 & 59306 & 0 & 0 & 0 & 7436 & 4238 & -19922 & 0 & 0 & 0 & 3381 & 13027 & -17589 & 0 & 0 & 0 & -4055 & -5249 & 873 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-6641 & 1712 & 0 & 0 & 0 & 0 & 14774 & -5438 & 0 & 0 & 0 & 0 & 291 & -1375 & 0 & 0 & 0 & 0 & -8424 & 5101 & 0 & 0 & 0 & 0 \\
-1413 & 2563 & 0 & 0 & 0 & 0 & -5438 & 15831 & 0 & 0 & 0 & 0 & 1750 & -9367 & 0 & 0 & 0 & 0 & 5101 & -9027 & 0 & 0 & 0 & 0 \\
0 & 0 & -971 & 4803 & 7436 & 0 & 0 & 0 & 7287 & 0 & 0 & 0 & 0 & 0 & -2161 & 0 & 0 & 0 & 0 & 0 & -4155 & 0 & 0 & 0 \\
0 & 0 & 0 & 7689 & 4238 & 0 & 0 & 0 & 0 & 47493 & 16314 & 0 & 0 & 0 & 0 & -28101 & -5249 & 0 & 0 & 0 & 0 & -27081 & -15303 & 0 \\
0 & 0 & 0 & -5137 & -19922 & 0 & 0 & 0 & 0 & 16314 & 44322 & 0 & 0 & 0 & 0 & 4126 & 873 & 0 & 0 & 0 & 0 & -15303 & -25273 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-5863 & -4342 & 0 & 0 & 0 & 0 & 291 & 1750 & 0 & 0 & 0 & 0 & 13800 & 3921 & 0 & 0 & 0 & 0 & -8228 & -1328 & 0 & 0 & 0 & 0 \\
-4342 & -6283 & 0 & 0 & 0 & 0 & -1375 & -9367 & 0 & 0 & 0 & 0 & 3921 & 14788 & 0 & 0 & 0 & 0 & 1797 & 862 & 0 & 0 & 0 & 0 \\
0 & 0 & -2892 & -3665 & 3381 & 0 & 0 & 0 & -2161 & 0 & 0 & 0 & 0 & 0 & 6807 & 0 & 0 & 0 & 0 & 0 & -1754 & 0 & 0 & 0 \\
0 & 0 & 0 & -18848 & 13027 & 0 & 0 & 0 & 0 & -28101 & 4126 & 0 & 0 & 0 & 0 & 44363 & -11763 & 0 & 0 & 0 & 0 & 2586 & -5390 & 0 \\
0 & 0 & 0 & 13027 & -17589 & 0 & 0 & 0 & 0 & -5249 & 873 & 0 & 0 & 0 & 0 & -11763 & 41401 & 0 & 0 & 0 & 0 & 3985 & -24685 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
291 & -1375 & 0 & 0 & 0 & 0 & -8424 & 5101 & 0 & 0 & 0 & 0 & -8228 & 1797 & 0 & 0 & 0 & 0 & 16362 & -5522 & 0 & 0 & 0 & 0 \\
1750 & -9367 & 0 & 0 & 0 & 0 & 5101 & -9027 & 0 & 0 & 0 & 0 & -1328 & 862 & 0 & 0 & 0 & 0 & -5522 & 17532 & 0 & 0 & 0 & 0 \\
0 & 0 & -2161 & -8469 & -4055 & 0 & 0 & 0 & -4155 & 0 & 0 & 0 & 0 & 0 & -1754 & 0 & 0 & 0 & 0 & 0 & 8070 & 0 & 0 & 0 \\
0 & 0 & 0 & -28101 & -5249 & 0 & 0 & 0 & 0 & -27081 & -15303 & 0 & 0 & 0 & 0 & 2586 & 3985 & 0 & 0 & 0 & 0 & 52596 & 16567 & 0 \\
0 & 0 & 0 & 4126 & 873 & 0 & 0 & 0 & 0 & -15303 & -25273 & 0 & 0 & 0 & 0 & -5390 & -24685 & 0 & 0 & 0 & 0 & 16567 & 49085 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]$$

It is worth noticing that the rows and columns of $\left[k'^{(e)}\right]$ corresponding to the nodal rotations about the local z-axis are all zero.

$$\{f'^{(e)}\} = \left(\left[N^{(e)}\right]_1\right)^T\{p^{(e)}\}_1 J_1 \cdot W_1 + \left(\left[N^{(e)}\right]_1\right)^T\{p^{(e)}\}_1 J_1 \cdot W_1 + \left(\left[N^{(e)}\right]_1\right)^T\{p^{(e)}\}_1 J_1 \cdot W_1 + \left(\left[N^{(e)}\right]_1\right)^T\{p^{(e)}\}_1 J_1 \cdot W_1$$

$$= [0.065\ \ -1.165\ \ -4.667\ \ 0.00\ \ 0.00\ \ 0.00\ \ 0.065\ \ -1.165\ \ -4.667\ \ 0.00\ \ 0.00\ \ 0.00\ \ 0.060\ \ -1.082\ \ -4.333\ \ 0.00\ \ 0.00\ \ 0.00\ \ 0.060\ \ -1.082\ \ -4.333\ \ 0.00\ \ 0.00\ \ 0.00]^T$$

The 24×24 coordinate transformation array, $[R^{(e)}]$, of the element can be established in block form as:

$$\left[R^{(e)}\right] = \begin{bmatrix} \left[r^{(e)}\right] & [0] & [0] & [0] \\ [0] & \left[r^{(e)}\right] & [0] & [0] \\ [0] & [0] & \left[r^{(e)}\right] & [0] \\ [0] & [0] & [0] & \left[r^{(e)}\right] \end{bmatrix}$$, where each block has dimensions 6×6 and

$$\left[r^{(e)}\right] = \begin{bmatrix} t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} & 0 & 0 & 0 \\ t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} & 0 & 0 & 0 \\ n_x^{(e)} & n_y^{(e)} & n_z^{(e)} & 0 & 0 & 0 \\ 0 & 0 & 0 & t_{1x}^{(e)} & t_{1y}^{(e)} & t_{1z}^{(e)} \\ 0 & 0 & 0 & t_{2x}^{(e)} & t_{2y}^{(e)} & t_{2z}^{(e)} \\ 0 & 0 & 0 & n_x^{(e)} & n_y^{(e)} & n_z^{(e)} \end{bmatrix}$$

$$= \begin{bmatrix} 0.9985 & -0.0538 & -0.0135 & 0 & 0 & 0 \\ 0.0555 & 0.9686 & 0.2422 & 0 & 0 & 0 \\ 0 & -0.2425 & 0.9701 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.9985 & -0.0538 & -0.0135 \\ 0 & 0 & 0 & 0.0555 & 0.9686 & 0.2422 \\ 0 & 0 & 0 & 0 & -0.2425 & 0.9701 \end{bmatrix}$$

Finally, we can use Equations (14.8.7) and (14.8.8) to establish the element stiffness matrix and equivalent nodal force/moment vector in the global coordinate system:

$$\left[k^{(e)}\right] = \left[R^{(e)}\right]^T \left[k'^{(e)}\right] \left[R^{(e)}\right]$$

$$= \begin{bmatrix}
12659 & 3909 & 977 & 0 & 0 & 0 & -6596 & -877 & -219 & 0 & 0 & 0 & -6345 & -4209 & -1052 & 0 & 0 & 0 & 282 & 1178 & 294 & 0 & 0 & 0 \\
3909 & 12251 & 1557 & -1684 & 1684 & 421 & 2155 & 2313 & 821 & 0 & 0 & 0 & -4209 & -5629 & -684 & 0 & 0 & 0 & -1854 & -8935 & -1693 & 0 & 0 & 0 \\
977 & 1557 & 6413 & 6737 & -6737 & -1684 & 539 & 821 & -766 & 0 & 0 & 0 & -1052 & -684 & -3063 & 0 & 0 & 0 & -464 & -1693 & -2584 & 0 & 0 & 0 \\
0 & -1684 & 6737 & 60589 & -11726 & -2931 & 0 & -1263 & 5053 & 7554 & -6464 & -1616 & 0 & 842 & -3369 & -17401 & 12628 & 3157 & 0 & 2105 & -8421 & -28075 & 5562 & 1391 \\
0 & 1687 & -6737 & -11726 & 57078 & 14269 & 0 & -1684 & 6737 & 2631 & -18623 & -4656 & 0 & -842 & 3369 & 12628 & -17916 & -4479 & 0 & 842 & -3369 & -3533 & 796 & 199 \\
0 & 421 & -1684 & -2931 & 14269 & 3567 & 0 & -421 & 1684 & 658 & -4656 & -1164 & 0 & -211 & 842 & 3157 & -4479 & -1120 & 0 & 211 & -842 & -883 & 199 & 50 \\
-6596 & 2155 & 539 & 0 & 0 & 0 & 14175 & -5186 & -1297 & 0 & 0 & 0 & 282 & -1854 & -464 & 0 & 0 & 0 & -7861 & 4886 & 1221 & 0 & 0 & 0 \\
-877 & 2313 & 821 & -1263 & -1684 & -421 & -5186 & 15892 & 2151 & 0 & 0 & 0 & 1178 & -8935 & -1693 & 0 & 0 & 0 & 4886 & -9270 & -1279 & 0 & 0 & 0 \\
-219 & 821 & -766 & 5053 & 6737 & 1684 & -1297 & 2151 & 7825 & 0 & 0 & 0 & 294 & -1693 & -2584 & 0 & 0 & 0 & 1221 & -1279 & -4475 & 0 & 0 & 0 \\
0 & 0 & 0 & 7554 & 2631 & 658 & 0 & 0 & 0 & 49290 & 15559 & 3890 & 0 & 0 & 0 & -28075 & -3533 & -883 & 0 & 0 & 0 & -28770 & -14657 & -3664 \\
0 & 0 & 0 & -6464 & -18623 & -4656 & 0 & 0 & 0 & 15559 & 40024 & 10006 & 0 & 0 & 0 & 5562 & 796 & 199 & 0 & 0 & 0 & -14657 & -22192 & -5549 \\
0 & 0 & 0 & -1616 & -4656 & -1164 & 0 & 0 & 0 & 3890 & 10006 & 2501 & 0 & 0 & 0 & 1391 & 199 & 50 & 0 & 0 & 0 & -3664 & -5549 & -1387 \\
-6345 & -4209 & -1052 & 0 & 0 & 0 & 282 & 1178 & 294 & 0 & 0 & 0 & 14238 & 3833 & 958 & 0 & 0 & 0 & -8174 & -802 & -200 & 0 & 0 & 0 \\
-4209 & -5629 & -684 & 842 & -842 & -211 & -1854 & -8935 & -1693 & 0 & 0 & 0 & 3833 & 13907 & 1775 & 0 & 0 & 0 & 2230 & 657 & 603 & 0 & 0 & 0 \\
-1052 & -684 & -3063 & -3369 & 3369 & 842 & -464 & -1693 & -2584 & 0 & 0 & 0 & 958 & 1775 & 7250 & 0 & 0 & 0 & 557 & 603 & -1603 & 0 & 0 & 0 \\
0 & 0 & 0 & -17401 & 12628 & 3157 & 0 & 0 & 0 & -28075 & 5562 & 1391 & 0 & 0 & 0 & 43051 & -11500 & -2875 & 0 & 0 & 0 & 2424 & -6690 & -1672 \\
0 & 0 & 0 & 12628 & -17916 & -4479 & 0 & 0 & 0 & -3533 & 796 & 199 & 0 & 0 & 0 & -11500 & 40201 & 10050 & 0 & 0 & 0 & 2405 & -23080 & -5770 \\
0 & 0 & 0 & 3157 & -4479 & -1120 & 0 & 0 & 0 & -883 & 199 & 50 & 0 & 0 & 0 & -2875 & 10050 & 2513 & 0 & 0 & 0 & 601 & -5770 & -1443 \\
282 & -1854 & -464 & 0 & 0 & 0 & -7861 & 4886 & 1221 & 0 & 0 & 0 & -8174 & 2230 & 557 & 0 & 0 & 0 & 15754 & -5262 & -1315 & 0 & 0 & 0 \\
1178 & -8935 & -1693 & 2105 & 842 & -211 & 4886 & -9270 & -1279 & 0 & 0 & 0 & -802 & 657 & 603 & 0 & 0 & 0 & -5262 & 17548 & 2369 & 0 & 0 & 0 \\
294 & -693 & -2584 & -8421 & -3369 & -842 & 1221 & -1279 & -4475 & 0 & 0 & 0 & -200 & 603 & -1603 & 0 & 0 & 0 & -1315 & 2369 & 862 & 0 & 0 & 0 \\
0 & 0 & 0 & -28075 & -3533 & -883 & 0 & 0 & 0 & -28770 & -14657 & -3664 & 0 & 0 & 0 & 2424 & 2405 & 601 & 0 & 0 & 0 & 54420 & 15785 & 3946 \\
0 & 0 & 0 & 5562 & 796 & 199 & 0 & 0 & 0 & -14657 & -22196 & -5549 & 0 & 0 & 0 & -6690 & -23080 & -5770 & 0 & 0 & 0 & 15785 & 44481 & 11120 \\
0 & 0 & 0 & 1391 & 199 & 50 & 0 & 0 & 0 & -3664 & -5549 & -1387 & 0 & 0 & 0 & -1672 & -5770 & 1443 & 0 & 0 & 0 & 3946 & 11120 & 2780
\end{bmatrix}$$

We can see that, for the global coordinate system, there are nonzero stiffness terms for all rows and columns of $[k^{(e)}]$.

$$\left\{f^{(e)}\right\} = \left[R^{(e)}\right]^T \left\{f'^{(e)}\right\}$$

$$= [0.00\ 0.00\ \ -4.810\ 0.00\ 0.00\ 0.00\ 0.00\ 0.00\ \ -4.810\ 0.00\ 0.00\ 0.00\ 0.00\ 0.00\ \ -4.467\ 0.00\ 0.00\ 0.00\ 0.00\ 0.00\ \ -4.467\ 0.00\ 0.00\ 0.00]^T$$

One may observe that $\{f^{(e)}\}$ has nonzero nodal forces along the global Z-axis only. This should come as no surprise, since the distributed loads (expressed in the global coordinate system) that are applied on our element only have a nonzero component along that same axis! ■

14.9 A "Clever" Way to Approximately Satisfy C^1 Continuity Requirements for Thin Shells—The Discrete Kirchhoff Formulation

We will now dedicate a section to the finite element analysis of thin planar shells. As mentioned in Section 14.1, the Kirchhoff-Love (K-L) shell theory, which applies to such structures, can be thought of as an extension (to shells) of the Euler-Bernoulli beam theory. In the K-L theory, the through-thickness shear strain components, γ_{xz} and γ_{yz}, vanish. This implies that *plane shell sections that are normal to the undeformed reference surface remain plane and normal to the reference surface in the deformed configuration.* In terms of the previously derived Equations (14.1.3e, f), the K-L theory leads to satisfaction of the following expressions.

$$\gamma_{yz} = 0 \rightarrow \frac{\partial u_{oz}}{\partial y} - \theta_2 = 0 \rightarrow \theta_2 = \frac{\partial u_{oz}}{\partial y} \tag{14.9.1a}$$

$$\gamma_{zx} = 0 \rightarrow \frac{\partial u_{oz}}{\partial x} - \theta_1 = 0 \rightarrow \theta_1 = \frac{\partial u_{oz}}{\partial x} \tag{14.9.1b}$$

If we plug Equations (14.9.1a,b) into Equations (14.1.4d–f), the curvature (flexural) generalized strains for thin shells are defined as follows.

$$\varphi_{11} = \frac{\partial \theta_1}{\partial x} = \frac{\partial^2 u_{oz}}{\partial x^2} \tag{14.9.2a}$$

$$\varphi_{22} = \frac{\partial \theta_2}{\partial y} = \frac{\partial^2 u_{oz}}{\partial y^2} \tag{14.9.2b}$$

$$\varphi_{12} = \frac{1}{2}\left(\frac{\partial \theta_2}{\partial x} + \frac{\partial \theta_1}{\partial y}\right) = \frac{\partial^2 u_{oz}}{\partial x \partial y} \tag{14.9.2c}$$

The above expressions for curvature components include second-order derivatives of the displacement field u_{oz}. Similar expressions can be established for the virtual curvatures $\bar{\varphi}_{11}$, $\bar{\varphi}_{22}$ and $\bar{\varphi}_{12}$, which will now include second-order derivatives of the virtual normal displacement field w_{oz}:

$$\bar{\varphi}_{11} = \frac{\partial w_{\theta 1}}{\partial x} = \frac{\partial^2 w_{oz}}{\partial x^2} \tag{14.9.3a}$$

$$\bar{\varphi}_{22} = \frac{\partial w_{\theta 2}}{\partial y} = \frac{\partial^2 w_{oz}}{\partial y^2} \tag{14.9.3b}$$

$$\bar{\varphi}_{12} = \frac{1}{2}\left(\frac{\partial w_{\theta 2}}{\partial x} + \frac{\partial w_{\theta 1}}{\partial y}\right) = \frac{\partial^2 w_{oz}}{\partial x \partial y} \tag{14.9.3c}$$

Equations (14.9.3a–c) are a consequence of the kinematic assumptions of the K-L theory, according to which the virtual generalized shear strains $\bar{\gamma}_{zx}$ and $\bar{\gamma}_{yz}$ must vanish:

$$\bar{\gamma}_{zx} = 0 \tag{14.9.4a}$$

$$\bar{\gamma}_{yz} = 0 \tag{14.9.4b}$$

The weak form for shells derived in Box (14.5.1) is also valid for the K-L shell theory. If we take into account Equations (14.9.1a,b) and (14.9.4a,b), we can understand that all the weak-form terms depending on the through-thickness shear strains must vanish! This, in turn, allows us to define a generalized strain vector, $\{\hat{\varepsilon}\} = \begin{bmatrix} \varepsilon_{ox} & \varepsilon_{oy} & \gamma_{oxy} & \varphi_{11} & \varphi_{22} & 2\varphi_{12} \end{bmatrix}^T$, which only contains the membrane and bending (curvature) components. Similarly, we can use a generalized stress vector $\{\hat{\sigma}\} = \begin{bmatrix} \hat{n}_{xx} & \hat{n}_{yy} & \hat{n}_{xy} & \hat{m}_{xx} & \hat{m}_{yy} & \hat{m}_{xy} \end{bmatrix}^T$, and a virtual generalized strain vector, $\{\bar{\varepsilon}\} = \begin{bmatrix} \bar{\varepsilon}_{ox} & \bar{\varepsilon}_{oy} & \bar{\gamma}_{oxy} & \bar{\varphi}_{11} & \bar{\varphi}_{22} & 2\bar{\varphi}_{12} \end{bmatrix}^T$. The remainder of this Section will provide an overview of the finite element implementation of the discrete Kirchhoff theory. We will specifically discuss the case of the discrete Kirchhoff quadrilateral (DKQ) element.

Before we move on with the derivations of the shape functions for the specific element, we need to establish a useful lemma in Box 14.9.1.

Box 14.9.1 Lemma—Partial Derivative of a Function along a Specific Direction

The partial derivative of any function $f(x, y)$, with respect to a direction defined by a unit vector $\{n\}$ is given by the following relation.

$$\frac{\partial f}{\partial n} = \left(\vec{\nabla} f\right) \bullet \vec{n} \tag{14.9.5}$$

For a planar problem, if $\vec{n} = \begin{Bmatrix} n_x \\ n_y \end{Bmatrix}$, then

$$\frac{\partial f}{\partial n} = \frac{\partial f}{\partial x} n_x + \frac{\partial f}{\partial y} n_y \tag{14.9.6}$$

We are now going to derive the shape functions for a four-node, DKQ element, as described in Batoz and Ben-Tahar (1982). As a starting point, we will consider an eight-node quadrilateral (8Q) element, whose sides in the physical (Cartesian) space are all straight lines, as shown in Figure 14.17. The four nodal points 1,2,3 and 4 are located at the corners of the quadrilateral. The remaining nodes 5,6,7 and 8 are the midpoints of the four straight sides of the quadrilateral in the physical space.

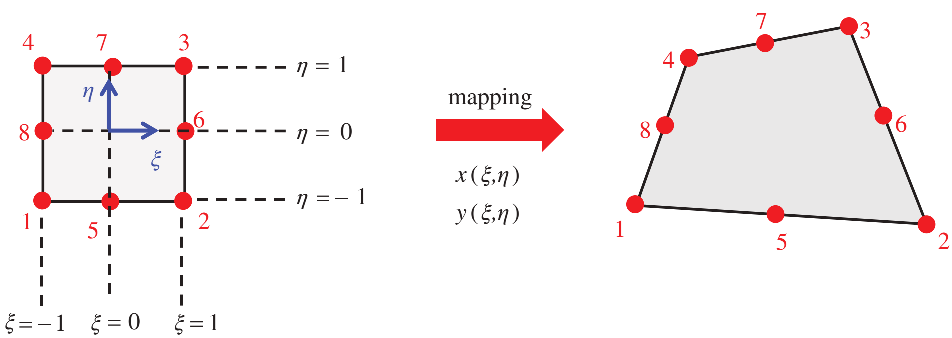


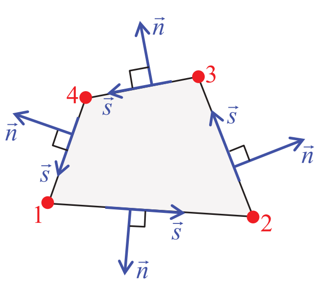
Figure 14.17 Isoparametric mapping for an 8Q element with straight sides in physical space.

For each of the element sides, we need to define the *unit normal outward vector*, $\vec{n}$, and the *unit tangential vector*, $\vec{s}$, as shown in Figure 14.18. The figure also provides the mathematical definition for the two vectors for each side, ij. The quantity $\ell_{ij}^{(e)}$ is merely the length of the line joining nodes i and j.

For the element sides, the rotations must be expressed in terms of components along the normal and tangential direction. The rotations about axis n and about axis s are related to the rotations θ_x, θ_y, in accordance with the following coordinate transformation equation (which is a special case of the general coordinate transformation rule in Section 7.7).

$$\begin{Bmatrix} \theta_n \\ \theta_n \end{Bmatrix} = \frac{1}{\ell_{ij}^{(e)}} \begin{bmatrix} -y_{ij}^{(e)} & x_{ij}^{(e)} \\ x_{ij}^{(e)} & y_{ij}^{(e)} \end{bmatrix} \begin{Bmatrix} \theta_x \\ \theta_x \end{Bmatrix} = \frac{1}{\ell_{ij}^{(e)}} \begin{bmatrix} -y_{ij}^{(e)} & -x_{ij}^{(e)} \\ x_{ij}^{(e)} & -y_{ij}^{(e)} \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_1 \end{Bmatrix} \tag{14.9.7}$$

We now come to the weak form for thin shells, which is the same as the expression in Box 14.5.1, except that we must remove the terms associated with the strains γ_{zx}, γ_{yz} and $\bar{\gamma}_{zx}$, $\bar{\gamma}_{yz}$. In accordance with our discussion above, the virtual work terms for flexure will now involve second-order derivatives of kinematic fields. This leads to the requirement for C^1 continuity for a finite element model, and C^1 continuity is impossible to strictly enforce in a conventional, multidimensional approximation with polynomial shape functions. This difficulty is circumvented in the DKQ formulation by an



For each side "*ij*" (e.g., side 12):

$$\vec{s} = \frac{1}{\ell_{ij}^{(e)}} \begin{Bmatrix} x_{ij}^{(e)} \\ y_{ij}^{(e)} \end{Bmatrix} = \frac{1}{\ell_{ij}^{(e)}} \begin{Bmatrix} x_j^{(e)} - x_i^{(e)} \\ y_j^{(e)} - y_i^{(e)} \end{Bmatrix}$$

$$\vec{n} = \frac{1}{\ell_{ij}^{(e)}} \begin{Bmatrix} y_{ij}^{(e)} \\ -x_{ij}^{(e)} \end{Bmatrix} = \frac{1}{\ell_{ij}^{(e)}} \begin{Bmatrix} y_j^{(e)} - y_i^{(e)} \\ x_i^{(e)} - x_j^{(e)} \end{Bmatrix}$$

Figure 14.18 Definition of unit normal outward and unit tangential vectors for element sides.

"approximate" enforcement of Equations (14.9.1a, b) and (14.9.4a, b), through stipulation that they only apply at specific locations in an element.

Let us establish the approximation of the DKQ element. We begin by stipulating the approximation for the rotation fields, $\theta_1^{(e)}$ and $\theta_2^{(e)}$, in the interior of the element:

$$\theta_1^{(e)}(\xi,\eta) = -\sum_{i=1}^{8}\left(N_i^{(8Q)}(\xi,\eta)\cdot\theta_{yi}^{(e)}\right) \tag{14.9.8a}$$

$$\theta_2^{(e)}(\xi,\eta) = \sum_{i=1}^{8}\left(N_i^{8Q}(\xi,\eta)\cdot\theta_{xi}^{(e)}\right) \tag{14.9.8b}$$

The eight shape functions in Equations (14.9.8a,b) are those of an 8Q serendipity element, which were presented in Section 6.8 and are also provided here:

$$N_1^{(8Q)}(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \tag{14.9.9a}$$

$$N_2^{(8Q)}(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \tag{14.9.9b}$$

$$N_3^{(8Q)}(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1) \tag{14.9.9c}$$

$$N_4^{(8Q)}(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \tag{14.9.9d}$$

$$N_5^{(8Q)}(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1-\eta) \tag{14.9.9e}$$

$$N_6^{(8Q)}(\xi,\eta) = \frac{1}{2}(1+\xi)\left(1-\eta^2\right) \tag{14.9.9f}$$

$$N_7^{(8Q)}(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1+\eta) \tag{14.9.9g}$$

$$N_8^{(8Q)}(\xi,\eta) = \frac{1}{2}(1-\xi)\left(1-\eta^2\right) \tag{14.9.9h}$$

As mentioned above, the intermediate nodes, 5,6,7,8, are the midpoints of the four sides of the quadrilateral element. Based on Figure 14.17, node 5 is located in the middle of side 12, node 6 is located in the middle of side 23, node 7 in the middle of 34, and node 8 in the middle of 41. We are going to stipulate that the following equations apply for the four intermediate nodal points:

$$\theta_{nk} = \left[\frac{\partial u_{oz}}{\partial s}\right]_{node\,k}, \quad \text{for} \quad k = 5,6,7,8 \tag{14.9.10a}$$

$$\theta_{sk} = \frac{1}{2}\left(\theta_{si}+\theta_{sj}\right), \quad \text{for} \quad k = 5,6,7,8 \tag{14.9.10b}$$

Equation (14.9.10a) is nothing other than the K-L hypothesis along the s-direction (i.e., the tangential direction) of each side of the element. On the other hand, Equation (14.9.10b) stipulates that the rotation about the side of the element at the middle is the average of the values at the two end points of the side. The specific equation relates the rotations of an intermediate node k to the corresponding rotations of the corner nodes i and j. The correspondence between a pair of nodes i and j and the node k

is provided in Table 14.9.1. Equation (14.9.10b) is also equivalent to the stipulation that the variation of the rotation in the direction s is linear along the element boundaries.

Next, we stipulate that *the variation of the transverse displacement u_{oz}, along each one of the boundary line segments* (i.e., the element sides) is determined with the expressions that we would be using for an Euler-Bernoulli beam element! For example, we can draw the side elevation of side (1-2) of our element, as shown in Figure 14.19, where we have also drawn the nodal transverse displacements and the rotations θ_n of the two end nodal points, 1 and 2. It is worth mentioning that this elevation with the nodal displacements/rotations is “reminiscent” of a planar beam segment! In fact, given the nodal displacements that we have drawn in the figure, and given that s corresponds to the axial coordinate along the beam, one can easily verify that Equation (14.9.10a) for $k = 5$ is nothing else that an Euler-Bernoulli beam statement (as described in Section 13.3) for the location of nodal point 5! Based on this observation, we will stipulate that the approximation of the normal displacement, $u_{oz}(s)$, along side (1-2) is given by the following expression.

$$u_{oz}(s) \approx N_{u1}^{(e)}(s) \cdot u_{z1}^{(e)} + N_{\theta 1}^{(e)}(s) \cdot \theta_{n1}^{(e)} + N_{u2}^{(e)}(s) \cdot u_{z2}^{(e)} + N_{\theta 2}^{(e)}(s) \cdot \theta_{n2}^{(e)} \tag{14.9.11}$$

The shape functions $N_i^{(e)}(s)$ in Equation (14.9.11) are those of a two-dimensional, two-node, Euler-Bernoulli beam element (Section 13.6) with a local coordinate system s-z, as shown in Figure 14.19. Thus, we have to establish the following Hermitian polynomials.

$$N_{u1}^{(e)}(s) = 1 - \frac{3s^2}{\left(\ell_{12}^{(e)}\right)^2} + \frac{2s^3}{\left(\ell_{12}^{(e)}\right)^3} \tag{14.9.12a}$$

Table 14.9.1 Correspondence between pair of nodes *i, j* and mid-side node *k*.

i	*j*	*k*
1	2	5
2	3	6
3	4	7
4	1	8

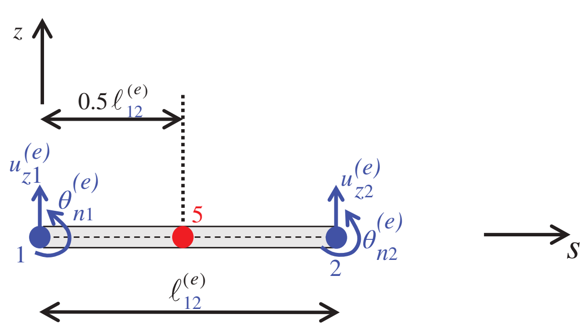


Figure 14.19 Side elevation of shell, for side (1-2).

$$N_{\theta 1}^{(e)}(s) = s - \frac{2s^2}{\ell_{12}^{(e)}} + \frac{s^3}{\left(\ell_{12}^{(e)}\right)^2} \tag{14.9.12b}$$

$$N_{u2}^{(e)}(s) = \frac{3s^2}{\left(\ell_{12}^{(e)}\right)^2} - \frac{2s^3}{\left(\ell_{12}^{(e)}\right)^3} \tag{14.9.12c}$$

$$N_{\theta 2}^{(e)}(s) = -\frac{s^2}{\ell_{12}^{(e)}} + \frac{s^3}{\left(\ell_{12}^{(e)}\right)^2} \tag{14.9.12d}$$

If we plug Equations (14.9.12a–d) into Equation (14.9.11) and then take the first derivative of the resulting expression with respect to the axial local coordinate, s, we obtain:

$$\frac{du_{oz}}{ds} \approx \left(\frac{6s}{\left(\ell_{12}^{(e)}\right)^2} - \frac{6s^2}{\left(\ell_{12}^{(e)}\right)^3} \right) \cdot \left(u_{z2}^{(e)} - u_{z1}^{(e)} \right) + \left(1 - \frac{4s}{\ell_{12}^{(e)}} - \frac{3s^2}{\left(\ell_{12}^{(e)}\right)^2} \right) \cdot \theta_{n1}^{(e)} + \left(-\frac{2s}{\ell_{12}^{(e)}} - \frac{3s^2}{\left(\ell_{12}^{(e)}\right)^2} \right) \cdot \theta_{n2}^{(e)} \tag{14.9.13}$$

Since node 5, $k = 5$, is located at the middle of side (12), we can set $s = 0.5\ell_{12}^{(e)}$ in Equation (14.9.13) to obtain:

$$\left[\frac{\partial u_{oz}}{\partial s} \right]_{node5} = \frac{3}{2\ell_{12}^{(e)}} \left(u_{z2}^{(e)} - u_{z1}^{(e)} \right) - \frac{1}{4} \left(\theta_{n1}^{(e)} + \theta_{n2}^{(e)} \right) \tag{14.9.14}$$

If we plug Equation (14.9.10a) for $k = 5$ into (14.9.14), we obtain:

$$\theta_{n5} = \frac{3}{2\ell_{12}^{(e)}} \left(u_{z2}^{(e)} - u_{z1}^{(e)} \right) - \frac{1}{4} \left(\theta_{n1}^{(e)} + \theta_{n2}^{(e)} \right) \tag{14.9.15}$$

Finally, if we use the first row of the matrix Equation (14.9.7) (i.e., the row giving θ_n), we can express the left-hand side of Equation (14.9.15) in terms of θ_{x5}, θ_{y5}:

$$-\frac{y_{12}^{(e)}}{\ell_{12}^{(e)}} \theta_{x5} + \frac{x_{12}^{(e)}}{\ell_{12}^{(e)}} \theta_{y5} = \frac{3}{2\ell_{12}^{(e)}} \left(u_{z2}^{(e)} - u_{z1}^{(e)} \right) + \frac{y_{12}^{(e)}}{4\ell_{12}^{(e)}} \left(\theta_{x1}^{(e)} + \theta_{x2}^{(e)} \right) - \frac{x_{12}^{(e)}}{4\ell_{12}^{(e)}} \left(\theta_{y1}^{(e)} + \theta_{y2}^{(e)} \right) \tag{14.9.16a}$$

If we plug $k = 5$ into Equation (14.9.10b) and also account for the second row of the matrix Equation (14.9.7) (i.e., the row giving θ_s), we obtain:

$$\frac{x_{12}^{(e)}}{\ell_{12}^{(e)}} \theta_{x5} + \frac{y_{12}^{(e)}}{\ell_{12}^{(e)}} \theta_{y5} = \frac{x_{12}^{(e)}}{2\ell_{12}^{(e)}} \left(\theta_{x1} + \theta_{x2} \right) + \frac{y_{12}^{(e)}}{2\ell_{12}^{(e)}} \left(\theta_{y1} + \theta_{y2} \right) \tag{14.9.16b}$$

Equations (14.9.16a) and (14.9.16b) can be rearranged into a new pair of equations, wherein θ_{x5}, θ_{y5} are expressed in terms of the normal displacements and in-plane rotations of nodes 1 and 2:

$$\theta_{x5} = -\frac{3y_{12}^{(e)}}{2\left(\ell_{12}^{(e)}\right)^2}\left(u_{z2}^{(e)} - u_{z1}^{(e)}\right) + \left[\frac{\left(x_{12}^{(e)}\right)^2}{2\left(\ell_{12}^{(e)}\right)^2} - \frac{\left(y_{12}^{(e)}\right)^2}{4\left(\ell_{12}^{(e)}\right)^2}\right]\left(\theta_{x1}^{(e)} + \theta_{x2}^{(e)}\right) + \frac{3x_{12}^{(e)}y_{12}^{(e)}}{4\left(\ell_{12}^{(e)}\right)^2}\left(\theta_{y1}^{(e)} + \theta_{y2}^{(e)}\right) \tag{14.9.17a}$$

$$\theta_{y5} = \frac{3x_{12}^{(e)}}{2\left(\ell_{12}^{(e)}\right)^2}\left(u_{z2}^{(e)} - u_{z1}^{(e)}\right) + \frac{3x_{12}^{(e)}y_{12}^{(e)}}{4\left(\ell_{12}^{(e)}\right)^2}\left(\theta_{x1}^{(e)} + \theta_{x2}^{(e)}\right) + \left[\frac{\left(y_{12}^{(e)}\right)^2}{2\left(\ell_{12}^{(e)}\right)^2} - \frac{\left(x_{12}^{(e)}\right)^2}{4\left(\ell_{12}^{(e)}\right)^2}\right]\left(\theta_{y1}^{(e)} + \theta_{y2}^{(e)}\right) \tag{14.9.17b}$$

Using the same exact procedure, we can express:

- Rotations θ_{x6} and θ_{y6} in terms of the normal displacements and in-plane rotations at nodes 2 and 3.
- Rotations θ_{x7} and θ_{y7} in terms of the normal displacements and in-plane rotations at nodes 3 and 4.
- Rotations θ_{x8} and θ_{y8} in terms of the normal displacements and in-plane rotations at nodes 4 and 1.

We can eventually obtain the following matrix equation.

$$\begin{Bmatrix} \theta_{x5}^{(e)} \\ \theta_{y5}^{(e)} \\ \theta_{x6}^{(e)} \\ \theta_{y6}^{(e)} \\ \theta_{x7}^{(e)} \\ \theta_{y7}^{(e)} \\ \theta_{x8}^{(e)} \\ \theta_{y8}^{(e)} \end{Bmatrix} = \begin{bmatrix} d_5^{(e)} & -e_5^{(e)} & b_5^{(e)} & -d_5^{(e)} & -e_5^{(e)} & b_5^{(e)} & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_5^{(e)} & b_5^{(e)} & -c_5^{(e)} & a_5^{(e)} & b_5^{(e)} & -c_5^{(e)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d_6^{(e)} & -e_6^{(e)} & b_6^{(e)} & -d_6^{(e)} & -e_6^{(e)} & b_6^{(e)} & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_6^{(e)} & b_6^{(e)} & -c_6^{(e)} & a_6^{(e)} & b_6^{(e)} & -c_6^{(e)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & d_7^{(e)} & -e_7^{(e)} & b_7^{(e)} & -d_7^{(e)} & -e_7^{(e)} & b_7^{(e)} \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_7^{(e)} & b_7^{(e)} & -c_7^{(e)} & a_7^{(e)} & b_7^{(e)} & -c_7^{(e)} \\ -d_8^{(e)} & -e_8^{(e)} & b_8^{(e)} & 0 & 0 & 0 & 0 & 0 & 0 & d_8^{(e)} & -e_8^{(e)} & b_8^{(e)} \\ a_8^{(e)} & b_8^{(e)} & -c_8^{(e)} & 0 & 0 & 0 & 0 & 0 & 0 & -a_8^{(e)} & b_8^{(e)} & -c_8^{(e)} \end{bmatrix} \begin{Bmatrix} u_{z1}^{(e)} \\ \theta_{x1}^{(e)} \\ \theta_{y1}^{(e)} \\ u_{z2}^{(e)} \\ \theta_{x2}^{(e)} \\ \theta_{y2}^{(e)} \\ u_{z3}^{(e)} \\ \theta_{x3}^{(e)} \\ \theta_{y3}^{(e)} \\ u_{z4}^{(e)} \\ \theta_{x4}^{(e)} \\ \theta_{y4}^{(e)} \end{Bmatrix} \tag{14.9.18}$$

where

$$a_k = \frac{3x_{ij}^{(e)}}{2\left(\ell_{ij}^{(e)}\right)^2} \tag{14.9.19a}$$

$$b_k = \frac{3x_{ij}^{(e)} y_{ij}^{(e)}}{4\left(\ell_{ij}^{(e)}\right)^2} \tag{14.9.19b}$$

$$c_k = \frac{\left(x_{ij}^{(e)}\right)^2}{4\left(\ell_{ij}^{(e)}\right)^2} - \frac{\left(y_{ij}^{(e)}\right)^2}{2\left(\ell_{ij}^{(e)}\right)^2} \tag{14.9.19c}$$

$$d_k = \frac{3y_{ij}^{(e)}}{2\left(\ell_{ij}^{(e)}\right)^2} \tag{14.9.19d}$$

$$e_k = -\frac{\left(x_{ij}^{(e)}\right)^2}{2\left(\ell_{ij}^{(e)}\right)^2} + \frac{\left(y_{ij}^{(e)}\right)^2}{4\left(\ell_{ij}^{(e)}\right)^2} \tag{14.9.19e}$$

and k is the midpoint of side ij, in accordance with Table 14.9.1.

In summary, if we know the normal displacements and in-plane rotations at the four corner nodes, we also know the in-plane rotations at the intermediate nodes.

If we plug Equation (14.9.18) into Equations (14.9.8a) and (14.9.8b), we obtain:

$$\begin{Bmatrix} \theta_1^{(e)}(\xi,\eta) \\ \theta_2^{(e)}(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} H_1^{1(e)} & H_2^{1(e)} & H_3^{1(e)} & H_4^{1(e)} & H_5^{1(e)} & H_6^{1(e)} & H_7^{1(e)} & H_8^{1(e)} & H_9^{1(e)} & H_{10}^{1(e)} & H_{11}^{1(e)} & H_{12}^{1(e)} \\ H_1^{2(e)} & H_2^{2(e)} & H_3^{2(e)} & H_4^{2(e)} & H_5^{2(e)} & H_6^{2(e)} & H_7^{2(e)} & H_8^{2(e)} & H_9^{2(e)} & H_{10}^{2(e)} & H_{11}^{2(e)} & H_{12}^{2(e)} \end{bmatrix} \left\{U_f^{(e)}\right\} \tag{14.9.20}$$

where $H_i^{1(e)}(\xi,\eta)$, $H_i^{2(e)}(\xi,\eta)$ constitute modified shape functions accounting for the discrete Kirchhoff kinematic assumptions and $\left\{U_f^{(e)}\right\}$ is a (12 × 1) column vector, containing the *flexural degrees of freedom of the four corner nodes*:

$$\left\{U_f^{(e)}\right\} = \left[u_{z1}^{(e)} \;\; \theta_{x1}^{(e)} \;\; \theta_{y1}^{(e)} \;\; u_{z2}^{(e)} \;\; \theta_{x2}^{(e)} \;\; \theta_{y2}^{(e)} \;\; u_{z3}^{(e)} \;\; \theta_{x3}^{(e)} \;\; \theta_{y3}^{(e)} \;\; u_{z4}^{(e)} \;\; \theta_{x4}^{(e)} \;\; \theta_{y4}^{(e)}\right]^T \tag{14.9.21}$$

The expressions for the modified shape functions of the first three columns of the shape function array in Equation (14.9.20) (i.e., the shape functions corresponding to the nodal values of corner node 1) can be written as follows.

$$H_1^{1(e)}(\xi,\eta) = a_5 \cdot N_5^{(8Q)}(\xi,\eta) - a_8 \cdot N_8^{(8Q)}(\xi,\eta) \tag{14.9.22a}$$

$$H_2^{1(e)}(\xi,\eta) = b_5 \cdot N_5^{(8Q)}(\xi,\eta) + b_8 \cdot N_8^{(8Q)}(\xi,\eta) \tag{14.9.22b}$$

$$H_3^{1(e)}(\xi,\eta) = N_1^{(8Q)}(\xi,\eta) - c_5 \cdot N_5^{(8Q)}(\xi,\eta) - c_8 \cdot N_8^{(8Q)}(\xi,\eta) \tag{14.9.22c}$$

$$H_2^{1(e)}(\xi,\eta) = d_5 \cdot N_5^{(8Q)}(\xi,\eta) - d_8 \cdot N_5^{(8Q)}(\xi,\eta) \tag{14.9.22d}$$

$$H_2^{2(e)}(\xi,\eta) = -N_1^{(8Q)}(\xi,\eta) + e_5 \cdot N_5^{(8Q)}(\xi,\eta) + e_8 \cdot N_8^{(8Q)}(\xi,\eta) \tag{14.9.22e}$$

$$H_3^{2(e)}(\xi,\eta) = -b_5 \cdot N_5^{(8Q)}(\xi,\eta) - b_8 \cdot N_8^{(8Q)}(\xi,\eta) = -H_2^{1(e)}(\xi,\eta) \tag{14.9.22f}$$

Along the same way, we can obtain the remaining shape functions of Equation (14.9.20), corresponding to corner nodes 2, 3, and 4. We need to keep in mind the following:

- For node 2, we must replace subscript 1 in the right-hand side of Equations (14.9.22c,e) by 2 and subscripts 5 and 8 in Equations (14.9.22a–f) by subscripts 6 and 5, respectively.
- For node 3, we must replace subscript 1 in the right-hand side of Equations (14.9.22c,e) by 3 and subscripts 5 and 8 in Equations (14.9.22a–f) by subscripts 7 and 6, respectively.
- For node 4, we must replace subscript 1 in the right-hand side of Equations (14.9.22c,e) by 4 and subscripts 5 and 8 in Equations (14.9.22a–f) by subscripts 8 and 7, respectively.

Remark 14.9.1: We started our derivations by stipulating the variation of θ_1, θ_2 in the interior of the element, and then stipulated the variation of u_{oz} along the boundary of the element (i.e., along the perimeter of the element). This is an example of a special case of multifield (mixed) element, called *hybrid element*, in which we stipulate the approximation of several field functions (in our case, the rotation fields θ_1 and θ_2) in the interior of the element, while for other field functions (in our case, the displacement field u_{oz}; see Equation (14.9.11)), we stipulate their approximate variation along the boundary. ■

Remark 14.9.2: The modified shape functions for DKQ elements in Equations (14.9.20) and (14.9.22a–f) only affect the bending (plate) behavior. For the in-plane membrane degrees of freedom, the finite element approximation of Equations (14.6.1a,b) still applies. Furthermore, it is worth mentioning that we no longer establish an approximation for the field u_{oz} in the interior of the element! We can now establish the complete finite element approximation for a four-node, DQK element:

$$\begin{Bmatrix} u_{ox}^{(e)}(\xi,\eta) \\ u_{oy}^{(e)}(\xi,\eta) \\ \theta_1^{(e)}(\xi,\eta) \\ \theta_2^{(e)}(\xi,\eta) \end{Bmatrix} = \left[N_1^{(e)}(\xi,\eta)\right]\left\{U_1^{(e)}\right\} + \left[N_2^{(e)}(\xi,\eta)\right]\left\{U_2^{(e)}\right\} + \ldots + \left[N_4^{(e)}(\xi,\eta)\right]\left\{U_4^{(e)}\right\} \rightarrow$$

$$\rightarrow \begin{Bmatrix} u_{ox}^{(e)}(\xi,\eta) \\ u_{oy}^{(e)}(\xi,\eta) \\ \theta_1^{(e)}(\xi,\eta) \\ \theta_2^{(e)}(\xi,\eta) \end{Bmatrix} = \left[\left[N_1^{(e)}(\xi,\eta)\right]\left[N_2^{(e)}(\xi,\eta)\right]\ldots\left[N_4^{(e)}(\xi,\eta)\right]\right] \begin{Bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \vdots \\ \left\{U_4^{(e)}\right\} \end{Bmatrix} \quad (14.9.23)$$

where each 4 × 6 subarray, $\left[N_I^{(e)}(\xi,\eta)\right]$, containing the shape functions for the contribution of node I to the approximate fields, is given by:

$$\left[N_I^{(e)}(\xi,\eta)\right]=\begin{bmatrix} N_I^{(e)}(\xi,\eta) & 0 & 0 & 0 & 0 & 0 \\ 0 & N_I^{(e)}(\xi,\eta) & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{3I-2}^{1(e)}(\xi,\eta) & H_{3I-1}^{1(e)}(\xi,\eta) & H_{3I}^{1(e)}(\xi,\eta) & 0 \\ 0 & 0 & H_{3I-2}^{2(e)}(\xi,\eta) & H_{3I-1}^{2(e)}(\xi,\eta) & H_{3I}^{2(e)}(\xi,\eta) & 0 \end{bmatrix} \qquad (14.9.24)$$

▮

We can also formulate the generalized strain-displacement relation for a DKQ element, accounting for the fact that we do not have through-thickness shear strains and by taking appropriate derivatives of the approximate fields. Specifically, we can write:

$$\left\{\hat{\varepsilon}^{(e)}\right\}=\left\{\begin{matrix}\left\{\hat{\varepsilon}_m^{(e)}\right\}\\ \left\{\hat{\varepsilon}_f^{(e)}\right\}\end{matrix}\right\}=\left[\begin{bmatrix}\left[B_{m1}^{(e)}(\xi,\eta)\right]\\ \left[B_{f1}^{(e)}(\xi,\eta)\right]\end{bmatrix}\begin{bmatrix}\left[B_{m1}^{(e)}(\xi,\eta)\right]\\ \left[B_{f1}^{(e)}(\xi,\eta)\right]\end{bmatrix}\cdots\begin{bmatrix}\left[B_{m4}^{(e)}(\xi,\eta)\right]\\ \left[B_{f4}^{(e)}(\xi,\eta)\right]\end{bmatrix}\right]\left\{\begin{matrix}\left\{U_1^{(e)}\right\}\\ \left\{U_2^{(e)}\right\}\\ \vdots\\ \left\{U_4^{(e)}\right\}\end{matrix}\right\}\rightarrow$$

$$\rightarrow\left\{\hat{\varepsilon}^{(e)}\right\}=\left[B^{(e)}(\xi,\eta)\right]\left\{U^{(e)}\right\} \qquad (14.9.25)$$

where $\left[B_{mI}^{(e)}(\xi,\eta)\right]$ is the membrane strain-displacement array for node I, given by Equation (14.6.10a), and $\left[B_{fI}^{(e)}(\xi,\eta)\right]$ is the flexural strain-displacement array for node I of a DKQ element, given by:

$$\left[B_{fI}^{(e)}(\xi,\eta)\right]=\begin{bmatrix} 0 & 0 & 0 & \dfrac{\partial H_{3I-2}^{1(e)}}{\partial x} & \dfrac{\partial H_{3I-1}^{1(e)}}{\partial x} & \dfrac{\partial H_{3I}^{1(e)}}{\partial x} \\ 0 & 0 & 0 & \dfrac{\partial H_{3I-2}^{2(e)}}{\partial y} & \dfrac{\partial H_{3I-1}^{2(e)}}{\partial y} & \dfrac{\partial H_{3I}^{2(e)}}{\partial y} \\ 0 & 0 & 0 & \dfrac{\partial H_{3I-2}^{1(e)}}{\partial y}+\dfrac{\partial H_{3I-2}^{2(e)}}{\partial x} & \dfrac{\partial H_{3I-1}^{1(e)}}{\partial y}+\dfrac{\partial H_{3I-1}^{2(e)}}{\partial x} & \dfrac{\partial H_{3I}^{1(e)}}{\partial y}+\dfrac{\partial H_{3I}^{2(e)}}{\partial x} \end{bmatrix} \qquad (14.9.26)$$

The $\left[\hat{D}^{(e)}\right]$ array of a DKQ element, giving the generalized stresses from the generalized strains, must only include the terms corresponding to membrane forces and the terms corresponding to flexural moments. In other words, the generalized stress-strain law for a DKQ element attains the following form.

$$\left\{\hat{\sigma}^{(e)}\right\}=\left\{\begin{matrix}\left\{\hat{\sigma}_m^{(e)}\right\}\\ \left\{\hat{\sigma}_f^{(e)}\right\}\end{matrix}\right\}=\left[\hat{D}^{(e)}\right]\left\{\hat{\varepsilon}^{(e)}\right\} \qquad (14.9.27)$$

For the special case of linear isotropic elasticity where the material parameters E and ν do not vary with z, we can use the arrays $\left[\hat{D}_m^{(e)}\right]$ (Equation 14.4.37) and $\left[\hat{D}_f^{(e)}\right]$ (Equation 14.4.29) to write:

$$\left[\hat{D}^{(e)}\right] = \begin{bmatrix} \left[\hat{D}_m^{(e)}\right] & [0] \\ [0] & \left[\hat{D}_f^{(e)}\right] \end{bmatrix} \tag{14.9.28}$$

Remark 14.9.3: The element stiffness matrix $[k^{(e)}]$ of a DKQ element can be obtained using Equation (14.6.13). ▮

Remark 14.9.4: The use of the DKQ shape functions of Equation (14.9.24) for calculation of the equivalent nodal force/moment vector $\{f^{(e)}\}$ may create theoretical difficulties for the quantities associated with plate behavior. To circumvent these difficulties, which will not be discussed here, $\{f^{(e)}\}$ can be obtained using the shape functions and equations of Section 14.6. ▮

Remark 14.9.5: The considerations in Remarks 14.9.3 and 14.9.4 provide the element stiffness matrix and equivalent nodal force/moment vector in the local coordinate system. The transformation equations presented in Section 14.8, which are still valid for DKQ elements, must be used to obtain the corresponding quantities in the global coordinate system. ▮

14.10 Continuum-Based Formulation for Nonplanar (Curved) Shells

The discussion for shells presented so far has been focused on planar (flat) shells. The finite element method can also be used for the analysis of curved shells, which are three-dimensional surfaces. We will rely on a continuum-based (CB) formulation, in accordance with descriptions in Hughes (2000). The CB formulation constitutes an extension (to shells) of the pertinent approach for beams, presented in Sections 13.12 and 13.13. If we go to a location of a shell in the physical space, which corresponds to given values of ξ and η, and draw many points with the same, given ξ and η values but with varying ζ-values, these points define (in the physical space) a curve called the *fiber* of the shell at the specific location. Any surface defined by the points with $\zeta = \zeta_o$ = fixed is called a *lamina*. For nonplanar shells, we need to modify (more accurately, to generalize!) the kinematic assumptions of shell theory. Specifically, we no longer stipulate that the normal stress along one of the physical axes is zero. *Instead, we assume that the normal stresses perpendicular to the lamina plane are zero.* This is the generalization of the statement that $\sigma_{zz} = 0$ for flat shells. *We also stipulate that the change in fiber length can be neglected* when we establish the displacement fields. This kinematic assumption is a generalization of the stipulation that we neglect the effect of thickness changes in planar shells.

We will obtain the formulation of a curved shell using the concept of an underlying continuum element, similar to the procedure provided in Section 13.13 for curved beams. Of course, instead of having the reference curve and top and bottom curves of beams, we will be having a (curved) reference surface and the top and bottom surfaces, as shown in Figure 14.20. Our description will rely on a four-node shell element, although a curved geometry might be better represented using more nodes—for example, eight nodes.

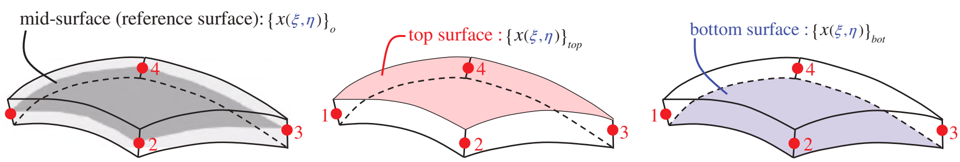


Figure 14.20 Reference, top, and bottom surfaces for curved shell.

Any three-dimensional surface is *parameterized* by two variables. This means that we only need a pair of parametric coordinates ξ, η to uniquely identify the physical coordinates of each point on the surface. The physical location of a point in a three-dimensional surface is established by a set of three Cartesian coordinates, $\{x\} = [x\ y\ z]^T$. As shown in Figure 14.20, the location of each point in the reference surface, the top surface and the bottom surface can be obtained in terms of ξ and η through a coordinate mapping. The element itself will be described by the nodal points that lie on the reference surface, as shown in Figure 14.20. In the derivations described herein, the reference surface will coincide with the mid-surface of the element. For each nodal point, we can define the fiber vector, as shown in Figure 14.21, if we know the physical coordinates of the top and bottom surface, for the same ξ and η values as those of the nodal point. Per Figure 14.21, the fiber vector of each node I is a unit vector parallel to the line that joins the top and bottom surface at the parametric location of node I.

For each node I of the shell element, the components of the unit vector in the fiber direction can be obtained from the following expression.

$$\vec{e}_f^{\,I} = \begin{Bmatrix} e_{fx}^I \\ e_{fy}^I \\ e_{fz}^I \end{Bmatrix} = \frac{1}{\sqrt{\left(x_{top}^I - x_{bot}^I\right)^2 + \left(y_{top}^I - y_{bot}^I\right)^2 + \left(z_{top}^I - z_{bot}^I\right)^2}} \begin{Bmatrix} x_{top}^I - x_{bot}^I \\ y_{top}^I - y_{bot}^I \\ z_{top}^I - z_{bot}^I \end{Bmatrix} \tag{14.10.1}$$

For given values of ξ, η, the distance between the top and bottom surfaces is the *thickness*, d, of the element. We now introduce a third parametric coordinate, ζ, which measures the location along the fiber. The value $\zeta = -1$ corresponds to the bottom

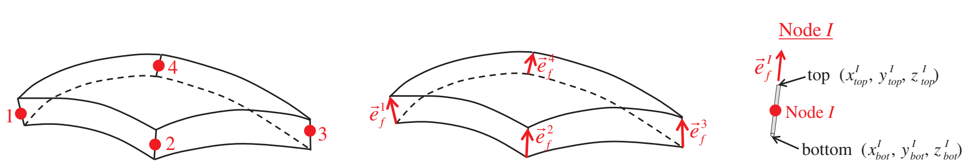


Figure 14.21 Fiber vector for each nodal point.

surface, the value $\zeta = 1$ corresponds to the top surface, and the value $\zeta = 0$ to the mid-surface. The physical coordinates of any point in the interior of the element can now be described by the following expression:

$$\{x(\xi,\eta,\zeta)\} = \{x(\xi,\eta)\}_o + \zeta\left[\frac{d(\xi,\eta)}{2}\vec{e}_f(\xi,\eta)\right] \tag{14.10.2}$$

where $\{x(\xi,\eta)\}_o$ is the vector with the three physical coordinates of the reference surface.

Equation (14.10.2) states that, for given parametric coordinates ξ and η, the location of any point can be found by detecting the location $\{x(\xi,\eta)\}_o$ at the reference surface, and then moving along the direction of the fiber by the amount $\zeta \cdot \dfrac{d(\xi,\eta)}{2}$.

One question that arises is how to determine the product of the thickness, d, by the fiber vector, $\vec{e}_f$, at any location (ξ,η). The answer is: in the very same way that we determine any quantity in the interior of the element—by interpolating between the nodal points through the shape functions! Thus, we can write:

$$d(\xi,\eta)\cdot\vec{e}_f(\xi,\eta) = \sum_{I=1}^{n}\left[N_I^{(e)}(\xi,\eta)\left(d_I\cdot\vec{e}_f^{\,I}\right)\right] \tag{14.10.3a}$$

where d_I is the thickness value at node I. Similarly, the physical coordinates for a location (ξ,η) of the reference surface can be found from the physical coordinates of the four nodal points as follows.

$$\{x(\xi,\eta)\}_o = \sum_{I=1}^{n}\left[N_I^{(e)}(\xi,\eta)\left\{x_I^{(e)}\right\}\right] \tag{14.10.3b}$$

Combining Equations (14.10.2) and (14.10.3a,b) eventually yields the following expression.

$$\{x(\xi,\eta,\zeta)\} = \sum_{I=1}^{n}\left[N_I^{(e)}(\xi,\eta)\left(\left\{x_I^{(e)}\right\} + \zeta\cdot\frac{d_I}{2}\cdot\vec{e}_f^{\,I}\right)\right] \tag{14.10.4}$$

Equation (14.10.4) constitutes a *mapping* from a three-dimensional parametric space (ξ, η, ζ) to the three-dimensional physical space (x, y, z).

The next step is to define the lamina, that is, the tangent plane to the shell reference surface in the element. It is known from considerations of differential geometry (e.g., O'Neill 2006) that, for a curved surface *parameterized* (i.e. described) by two coordinates ξ, η, the tangent plane at any location P on the surface is the plane defined by two tangential unit vectors, $\vec{e}_\xi$ and $\vec{e}_\eta$, schematically shown in Figure 14.22 and defined by the following two expressions.

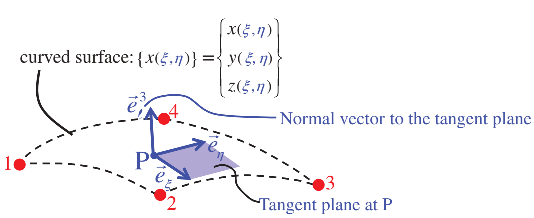


Figure 14.22 Tangent vectors, tangent plane, and normal vector to lamina.

$$\vec{e}_\xi = \frac{\{x_\xi\}}{\left\|\{x_\xi\}\right\|} \tag{14.10.5a}$$

$$\vec{e}_\eta = \frac{\{x_\eta\}}{\left\|\{x_\eta\}\right\|} \tag{14.10.5b}$$

where

$$\{x_\xi\} = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \end{bmatrix}^T \tag{14.10.6a}$$

$$\{x_\eta\} = \begin{bmatrix} \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix}^T \tag{14.10.6b}$$

Next, we will define a set of orthogonal unit vectors $\vec{e}_\ell^{\,1}, \vec{e}_\ell^{\,2}, \vec{e}_\ell^{\,3}$, describing *the local coordinate axes of a lamina*. We begin by establishing the laminar vector $\vec{e}_\ell^{\,3}$, which, as shown in Figure 14.22, is normal to the tangent plane and can be computed using the following expression.

$$\vec{e}_\ell^{\,3} = \frac{\vec{e}_\xi \times \vec{e}_\eta}{\left\|\vec{e}_\xi \times \vec{e}_\eta\right\|} \tag{14.10.7}$$

Next, we define the following two auxiliary vectors.

$$\vec{e}_a = \frac{\vec{e}_\xi + \vec{e}_\eta}{\left\|\vec{e}_\xi + \vec{e}_\eta\right\|} \tag{14.10.8a}$$

$$\vec{e}_b = \frac{\vec{e}_\ell^{\,3} \times \vec{e}_a}{\left\|\vec{e}_\ell^{\,3} \times \vec{e}_a\right\|} \tag{14.10.8b}$$

Finally, the two *laminar basis unit vectors* $\vec{e}_\ell^{\,1}, \vec{e}_\ell^{\,2}$ can be established using the equations:

$$\vec{e}_\ell^{\,1} = \frac{\sqrt{2}}{2}\left(\vec{e}_a - \vec{e}_b\right) \tag{14.10.9a}$$

$$\vec{e}_\ell^{\,2} = \frac{\sqrt{2}}{2}\left(\vec{e}_a + \vec{e}_b\right) \tag{14.10.9b}$$

Remark 14.10.1: The laminar basis vectors $\vec{e}_\ell^{\,1}, \vec{e}_\ell^{\,2}$ lie in the tangent plane to the curved surface. For a general curved surface, this means that each of these two laminar basis vectors is a linear combination of $\vec{e}_\xi$ and $\vec{e}_\eta$. ▮

We will now establish the considerations of the CB element. Specifically, as shown in Figure 14.23, we will consider an underlying, three-dimensional continuum element, whose nodes are the endpoints of the fibers at the nodal points of the shell element. The figure specifically describes the case where the shell element has four nodes; in this case, the underlying continuum element is an eight-node hexahedral (8H) one. As also shown in the figure, the motion (nodal displacements) of the nodes of the continuum element will

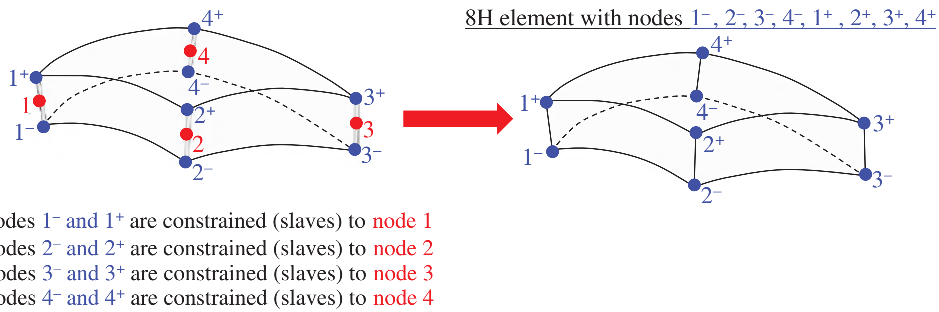


Figure 14.23 Underlying continuum element for four-node, curved shell element.

be *constrained* to the motion (nodal displacements and rotations) of the shell element. This is why the specific formulation is sometimes referred to as a *constrained continuum approach*. The constraints are essentially established by regarding the fibers as rigid bars, connecting the top and the bottom surface of the shell. For each node I ($I = 1, 2, 3, 4$) of the shell element, we have a corresponding pair of nodes, I^- and I^+.

The computations for the underlying continuum element can be conducted in accordance with the standard methodology for three-dimensional isoparametric elements, presented in Chapter 9. The coordinate mapping is described by Equation (14.10.4). The mapping considered for the underlying continuum element is also schematically shown in Figure 14.24.

The displacements of the underlying continuum element can be obtained from the displacements of the nodal points in the shell element, using rigid-body kinematic constraint equations in accordance with Appendix D. If we know the translations and rotations for node I of the shell reference surface, then the translations of the top surface node I^+ can be obtained from the following equation, which is *based on the assumption that node I^+ is connected by a rigid bar* (i.e., the fiber) *to node I* and the position vector of node I^+ with respect to node I is equal to $\frac{d_I}{2}\vec{e}_f^{\,I}$:

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}_{I^+} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}_{I} + \frac{d_I}{2}\begin{bmatrix} 0 & e_{fz}^I & -e_{fy}^I \\ -e_{fz}^I & 0 & e_{fx}^I \\ e_{fy}^I & -e_{fx}^I & 0 \end{bmatrix}\begin{Bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix}_{I} \tag{14.10.10}$$

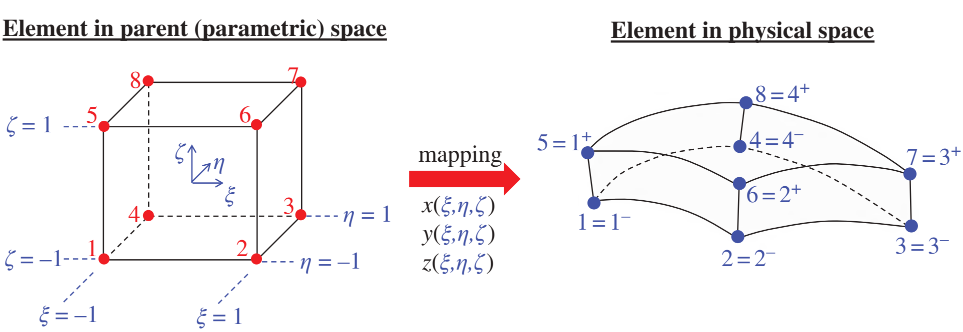


Figure 14.24 Mapping for underlying continuum element of a four-node, curved shell element.

Equation (14.10.10) can also be cast in the following matrix transformation form.

$$\{U_{I^+}\} = [T_{I^+}]\{U_I\} \tag{14.10.11}$$

where

$$\{U_{I^+}\} = \begin{bmatrix} u_{xI^+} & u_{yI^+} & u_{zI^+} \end{bmatrix}^T \tag{14.10.11a}$$

$$\{U_I\} = \begin{bmatrix} u_{xI} & u_{yI} & u_{zI} & \theta_{xI} & \theta_{yI} & \theta_{zI} \end{bmatrix}^T \tag{14.10.11b}$$

and

$$[T_{I^+}] = \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{d_I}{2}e^I_{fz} & -\frac{d_I}{2}e^I_{fy} \\ 0 & 1 & 0 & -\frac{d_I}{2}e^I_{fz} & 0 & \frac{d_I}{2}e^I_{fx} \\ 0 & 0 & 1 & \frac{d_I}{2}e^I_{fy} & -\frac{d_I}{2}e^I_{fx} & 0 \end{bmatrix} \tag{14.10.11c}$$

Similarly, the translations of node I^- of the bottom surface can be obtained as follows.

$$\begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}_{I^-} = \begin{Bmatrix} u_x \\ u_y \\ u_z \end{Bmatrix}_I - \frac{d_I}{2}\begin{bmatrix} 0 & e^I_{fz} & -e^I_{fy} \\ -e^I_{fz} & 0 & e^I_{fx} \\ e^I_{fy} & -e^I_{fx} & 0 \end{bmatrix}\begin{Bmatrix} \theta_x \\ \theta_y \\ \theta_z \end{Bmatrix}_I \tag{14.10.12}$$

We can once again write:

$$\{U_{I^-}\} = [T_{I^-}]\{U_I\} \tag{14.10.13}$$

where

$$\{U_{I^-}\} = \begin{bmatrix} u_{xI^-} & u_{yI^-} & u_{zI^-} \end{bmatrix}^T \tag{14.10.14a}$$

and

$$[T_{I^-}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{d_I}{2}e^I_{fz} & \frac{d_I}{2}e^I_{fy} \\ 0 & 1 & 0 & \frac{d_I}{2}e^I_{fz} & 0 & -\frac{d_I}{2}e^I_{fx} \\ 0 & 0 & 1 & -\frac{d_I}{2}e^I_{fy} & \frac{d_I}{2}e^I_{fx} & 0 \end{bmatrix} \tag{14.10.14b}$$

Next, we define two vectors with nodal values. One vector, $\{U^{(e)}\}$, will contain the nodal displacements and rotations of the shell element (in our case, of the four nodes of the reference surface). The other vector, $\{U^{(8Q)}\}$, contains the nodal displacements of the underlying continuum element. The two vectors are defined by the following block expressions.

$$\left\{U^{(e)}\right\} = \begin{bmatrix} \left\{U_1^{(e)}\right\} \\ \left\{U_2^{(e)}\right\} \\ \left\{U_3^{(e)}\right\} \\ \left\{U_4^{(e)}\right\} \end{bmatrix} \tag{14.10.15a}$$

$$\left\{U^{(8H)}\right\} = \begin{bmatrix} \left\{U_{1^-}^{(e)}\right\} \\ \left\{U_{2^-}^{(e)}\right\} \\ \left\{U_{3^-}^{(e)}\right\} \\ \left\{U_{4^-}^{(e)}\right\} \\ \left\{U_{1^+}^{(e)}\right\} \\ \left\{U_{2^+}^{(e)}\right\} \\ \left\{U_{3^+}^{(e)}\right\} \\ \left\{U_{4^+}^{(e)}\right\} \end{bmatrix} \tag{14.10.15b}$$

Now, we can combine the transformation equations for all the nodal points of the shell element and of the underlying (8H) element into a single constraint matrix equation:

$$\left\{U^{(8H)}\right\} = [T]\left\{U^{(e)}\right\} \tag{14.10.16}$$

The transformation matrix [T] in Equation (14.10.16) can be written in block form as follows:

$$[T] = \begin{bmatrix} [T_{1^-}] & [0] & [0] & [0] \\ [0] & [T_{2^-}] & [0] & [0] \\ [0] & [0] & [T_{3^-}] & [0] \\ [0] & [0] & [0] & [T_{4^-}] \\ [T_{1^+}] & [0] & [0] & [0] \\ [0] & [T_{2^+}] & [0] & [0] \\ [0] & [0] & [T_{3^+}] & [0] \\ [0] & [0] & [0] & [T_{4^+}] \end{bmatrix} \tag{14.10.17}$$

Each of the block matrices in Equation (14.10.17) has dimensions (3 × 6).

There is one final consideration that we need to make—that is, how to apply the stress-strain law in a CB shell element: the quasi-plane stress condition must be enforced in the laminar coordinate system. Specifically, the normal stress component $\sigma_{33}^{(\ell)}$ along the third laminar local direction must be equal to zero for a curved shell. In accordance with Section 14.4, this allows us to establish a reduced constitutive equation for five strain components. If we write a general linearly elastic stress-strain law for the local laminar coordinate system, we have:

$$\left\{\sigma^{(\ell)}\right\} = \left[D^{(\ell)}\right]\left\{\varepsilon^{(\ell)}\right\} \tag{14.10.18}$$

where

$$\left\{\sigma^{(\ell)}\right\} = \left[\sigma_{11}^{(\ell)} \;\; \sigma_{22}^{(\ell)} \;\; \sigma_{33}^{(\ell)} \;\; \sigma_{12}^{(\ell)} \;\; \sigma_{23}^{(\ell)} \;\; \sigma_{31}^{(\ell)}\right]^T \tag{14.10.19a}$$

$$\left\{\varepsilon^{(\ell)}\right\} = \left[\varepsilon_{11}^{(\ell)} \ \ \varepsilon_{22}^{(\ell)} \ \ \varepsilon_{33}^{(\ell)} \ \ 2\varepsilon_{12}^{(\ell)} \ \ 2\varepsilon_{23}^{(\ell)} \ \ 2\varepsilon_{31}^{(\ell)}\right]^T \tag{14.10.19b}$$

Just like we did in Section 14.4, we will establish a reduced stress-strain relation in the form:

$$\left\{\tilde{\sigma}^{(\ell)}\right\} = \left[D_{red}^{(\ell)}\right]\left\{\varepsilon^{(\ell)}\right\} \tag{14.10.20}$$

where $\left\{\tilde{\sigma}^{(\ell)}\right\} = \left[\sigma_{11}^{(\ell)} \ \ \sigma_{22}^{(\ell)} \ \ \sigma_{12}^{(\ell)} \ \ \sigma_{23}^{(\ell)} \ \ \sigma_{31}^{(\ell)}\right]^T$ contains the stress components except for $\sigma_{33}^{(\ell)} = 0$, and $\left[D_{red}^{(\ell)}\right]$ is the material stiffness matrix obtained if we erase the third row of $[D^{(\ell)}]$. We can eventually reach an equation of the form:

$$\left\{\tilde{\sigma}^{(\ell)}\right\} = \left[\tilde{D}^{(\ell)}\right]\left\{\tilde{\varepsilon}^{(\ell)}\right\} \tag{14.10.21}$$

where, per Section 14.4, $\left[\tilde{D}^{(\ell)}\right]$ is a (5 × 5), reduced (*condensed*) material stiffness matrix. We can directly write the following equation for the components of $\left[\tilde{D}^{(\ell)}\right]$.

$$\tilde{D}_{ij}^{(\ell)} = D_{ij}^{(\ell)} - D_{i3}^{(\ell)}\frac{D_{3j}^{(\ell)}}{D_{33}^{(\ell)}} \tag{14.10.22}$$

To use the stress-strain law in the lamina coordinate system, we need to express the strains and the displacements in this system. To this end, a 3 × 3 transformation matrix, $[q]$, is constructed, which allows the transformation of vectors from the global coordinate system to the lamina coordinate system. The rows of matrix $[q]$ are simply equal to the unit vectors of the lamina coordinate system:

$$[q] = \left[\vec{e}_\ell^{\,1} \ \ \vec{e}_\ell^{\,2} \ \ \vec{e}_\ell^{\,3}\right]^T \tag{14.10.23}$$

Each displacement component in direction i of the laminar coordinate system is obtained from the displacement vector components in the global coordinate system, through the following transformation equation.

$$u_i^{(l)} = \sum_{m=1}^{3}\left(q_{im} \cdot u_m\right) \tag{14.10.24}$$

We can then symbolically write the following expression, giving the nodal displacements of the (8*H*) element in the laminar coordinate system, from the nodal displacements of the same element in the global coordinate system:

$$\left\{U_{(\ell)}^{(8H)}\right\} = \left[Q^{(e)}\right]\left\{U^{(8H)}\right\} \tag{14.10.25}$$

where $\left\{U_{(\ell)}^{(8H)}\right\}$ contains the nodal displacements of the underlying continuum element expressed in the *local lamina* coordinate system, and the transformation matrix $[Q^{(e)}]$ can be written in block form as follows.

$$\left[Q^{(e)}\right] = \begin{bmatrix} [q] & [0] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [q] & [0] & [0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [q] & [0] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [q] & [0] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [q] & [0] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [q] & [0] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [q] & [0] \\ [0] & [0] & [0] & [0] & [0] & [0] & [0] & [q] \end{bmatrix} \tag{14.10.26}$$

Remark 14.10.2: Since the laminar basis vectors generally vary inside a curved shell, the values of $\left\{U_{(\ell)}^{(8H)}\right\}$ are different at each location in the element! ▪

We can now go ahead, and define a matrix strain-displacement relation for the laminar local coordinate system:

$$\left\{\tilde{\varepsilon}^{(\ell)}\right\} = \left[B_{(\ell)}^{(e)}\right]\left\{U_{(\ell)}^{(8H)}\right\} \tag{14.10.27}$$

where the local laminar strain-displacement matrix, $\left[B_{(\ell)}^{(e)}\right]$, is given by the following expression.

$$\left[B_{(\ell)}^{(e)}\right] = \begin{bmatrix} \frac{\partial N_1^{(8H)}}{\partial x_1^{(\ell)}} & 0 & 0 & \frac{\partial N_2^{(8H)}}{\partial x_1^{(\ell)}} & 0 & 0 & \cdots & \frac{\partial N_8^{(8H)}}{\partial x_1^{(\ell)}} & 0 & 0 \\ 0 & \frac{\partial N_1^{(8H)}}{\partial x_2^{(\ell)}} & 0 & 0 & \frac{\partial N_2^{(8H)}}{\partial x_2^{(\ell)}} & 0 & \cdots & 0 & \frac{\partial N_8^{(8H)}}{\partial x_2^{(\ell)}} & 0 \\ \frac{\partial N_1^{(8H)}}{\partial x_2^{(\ell)}} & \frac{\partial N_1^{(8H)}}{\partial x_1^{(\ell)}} & 0 & \frac{\partial N_2^{(8H)}}{\partial x_2^{(\ell)}} & \frac{\partial N_2^{(8H)}}{\partial x_1^{(\ell)}} & 0 & \cdots & \frac{\partial N_8^{(8H)}}{\partial x_2^{(\ell)}} & \frac{\partial N_8^{(8H)}}{\partial x_1^{(\ell)}} & 0 \\ 0 & \frac{\partial N_1^{(8H)}}{\partial x_3^{(\ell)}} & \frac{\partial N_1^{(8H)}}{\partial x_2^{(\ell)}} & 0 & \frac{\partial N_2^{(8H)}}{\partial x_3^{(\ell)}} & \frac{\partial N_2^{(8H)}}{\partial x_2^{(\ell)}} & \cdots & 0 & \frac{\partial N_8^{(8H)}}{\partial x_3^{(\ell)}} & \frac{\partial N_8^{(8H)}}{\partial x_2^{(\ell)}} \\ \frac{\partial N_1^{(8H)}}{\partial x_3^{(\ell)}} & 0 & \frac{\partial N_1^{(8H)}}{\partial x_1^{(\ell)}} & \frac{\partial N_2^{(8H)}}{\partial x_3^{(\ell)}} & 0 & \frac{\partial N_2^{(8H)}}{\partial x_1^{(\ell)}} & \cdots & \frac{\partial N_8^{(8H)}}{\partial x_3^{(\ell)}} & 0 & \frac{\partial N_8^{(8H)}}{\partial x_1^{(\ell)}} \end{bmatrix} \tag{14.10.28}$$

The matrix $\left[B_{(\ell)}^{(e)}\right]$ contains derivatives of the shape functions with respect to the local laminar coordinates. The partial derivative of each shape function $N_a^{(8H)}$ with respect to the laminar coordinate $x_j^{(\ell)}$ can be obtained by means of the chain rule of differentiation, in terms of the corresponding partial derivatives with respect to the global coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$:

$$\frac{\partial N_a^{(8H)}}{\partial x_j^{(\ell)}} = \frac{\partial N_a^{(8H)}}{\partial x_1}\frac{\partial x_1}{\partial x_j^{(\ell)}} + \frac{\partial N_a^{(8H)}}{\partial x_2}\frac{\partial x_2}{\partial x_j^{(\ell)}} + \frac{\partial N_a^{(8H)}}{\partial x_3}\frac{\partial x_3}{\partial x_j^{(\ell)}} \tag{14.10.29}$$

The spatial coordinates (x_1, x_2, x_3) and $\left(x_1^{(\ell)}, x_2^{(\ell)}, x_3^{(\ell)}\right)$ are components of a position vector; thus, they follow the same coordinate transformation rule as the displacements in Equation (14.10.24). We can eventually obtain that the partial derivatives of laminar coordinates with respect to global coordinates are given by:

$$\frac{\partial x_i}{\partial x_j^{(\ell)}} = q_{ji} \tag{14.10.30}$$

If we plug Equation (14.10.30) into Equation (14.10.29), we obtain:

$$\frac{\partial N_a^{(8H)}}{\partial x_j^{(\ell)}} = \frac{\partial N_a^{(8H)}}{\partial x_1}q_{j1} + \frac{\partial N_a^{(8H)}}{\partial x_2}q_{j2} + \frac{\partial N_a^{(8H)}}{\partial x_3}q_{j3} = \sum_{m=1}^{3}\left(\frac{\partial N_a^{(8H)}}{\partial x_m}q_{jm}\right) \tag{14.10.31}$$

We can now establish an algorithm for the calculation of the stiffness matrix and nodal force/moment vector of a continuum-based, curved shell element. This procedure is summarized in Box 14.10.1, for the case that we use Gaussian quadrature with N_g quadrature points for the underlying continuum element.

Box 14.10.1 Calculation of Stiffness Matrix and Nodal Force Vector for Curved, CB Shell Element

1) For each quadrature point g, $g = 1, 2, \ldots, N_g$ of the underlying continuum element:
   a) Calculate the laminar coordinate system, the matrix $[q_g]$, the transformation matrix $\left[Q_g^{(e)}\right]$, the shape function array $\left[N_g^{(8H)}\right]$ and the matrix $\left[B_{(\ell)g}^{(e)}\right]$. The latter is obtained after we have calculated $\left[N_{x,g}^{(8H)}\right]$, containing the derivatives of the shape functions with respect to x, y, z, evaluated at the location of point g, using the procedures described in Section 9.3.1.
   b) Calculate the stiffness contribution of the quadrature point to the stiffness matrix of the continuum element, expressed in the global coordinate system:

$$\left[Q_g^{(e)}\right]^T\left[B_{(\ell)g}^{(e)}\right]^T\left[\widetilde{D}_g^{(\ell)}\right]\left[B_{(\ell)g}^{(e)}\right]\left[Q_g^{(e)}\right]J_gW_g$$

   c) Calculate the nodal force contribution of the quadrature point to the equivalent nodal force vector of the continuum element, expressed in the global coordinate system:

$$\left[N_g^{(e)}\right]^T\left\{b_g^{(e)}\right\}J_gW_g$$

2) Calculate the stiffness matrix of the underlying continuum element, by summing the contributions of all the quadrature points. If we have a total of N_g quadrature points, we obtain:

$$\left[k^{(8H)}\right] = \sum_{g=1}^{N_g} \left(\left[Q_g^{(e)}\right]^T \left[B_{(\ell)g}^{(e)}\right]^T \left[\tilde{D}_g^{(\ell)}\right] \left[B_{(\ell)g}^{(e)}\right] \left[Q_g^{(e)}\right] J_g W_g \right)$$

3) Calculate the equivalent nodal force vector due to body forces, $\left\{f_\Omega^{(8H)}\right\}$, of the underlying continuum element, by summing the contributions of all the quadrature points. If we have a total of N_g quadrature points, we obtain:

$$\left\{f_\Omega^{(8H)}\right\} = \sum_{g=1}^{N_g} \left(\left[N_g^{(e)}\right]^T \left\{b_g^{(e)}\right\} J_g W_g \right)$$

4) Calculate the equivalent nodal force vector $\left\{f_\Gamma^{(8H)}\right\}$ due to prescribed tractions (if any) on the boundary surfaces of the underlying continuum element, using the procedure described in Section 9.3.3.
5) Calculate the total equivalent nodal force vector of the underlying continuum element, using the equation: $\left\{f^{(8H)}\right\} = \left\{f_\Omega^{(8H)}\right\} + \left\{f_\Gamma^{(8H)}\right\}$
6) Using the transformation matrix $[T]$, given by Equation (14.10.17), which expresses the relation between the nodal displacements of the continuum element and the displacements and rotations of the nodes in the shell element, obtain the stiffness matrix, $[k^{(e)}]$, and the equivalent nodal force/moment vector $\{f^{(e)}\}$, of the shell element e:

$$\left[k^{(e)}\right] = [T]^T \left[k^{(8H)}\right] [T] \qquad \left\{f^{(e)}\right\} = [T]^T \left\{f^{(8H)}\right\}$$

Remark 14.10.3: The algorithm in Box 14.10.1 gives the stiffness matrix and nodal force vector in the global coordinate system. ▮

Remark 14.10.4: For a shell element, we commonly state that we arrange the quadrature points in *stacks*. A stack of quadrature points corresponds to a set of quadrature points having the same values of ξ and η. Thus, if we use a conventional $2 \times 2 \times 2$ quadrature rule for the underlying continuum element, as shown in Figure 14.25, we will be having a total of four stacks of quadrature points (each stack having two points, one corresponding to $\zeta = -1/\sqrt{3}$ and the other to $\zeta = 1/\sqrt{3}$). In practice, the quadrature points are numbered by stack. Thus, for Figure 14.25, we would have quadrature stacks 1, 2, 3, and 4, and for each stack we would be having quadrature points 1 and 2. The number of quadrature points per stack is usually an input parameter for a finite element model. For linearly elastic analysis, two-point stacks are expected to give good results. For nonlinear material behavior (e.g., elastoplastic materials, cracking materials, etc.), at least five or six quadrature points per stack are required to obtain accurate results. ▮

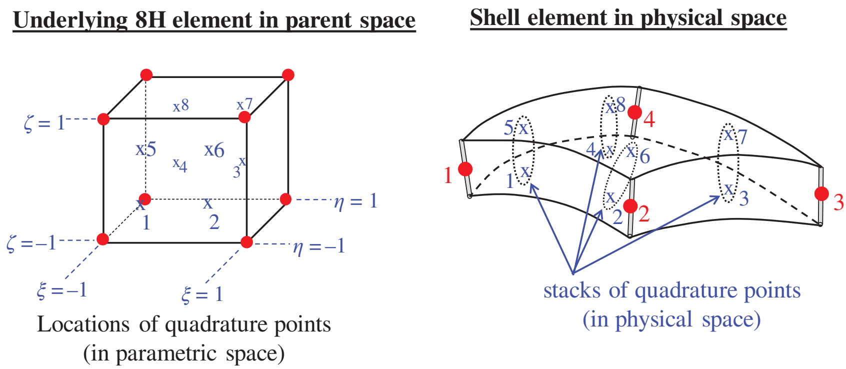
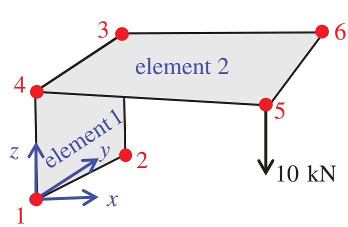


Figure 14.25 Stacks of quadrature points for a curved shell element.

Remark 14.10.5: As a follow-up note to Remark 14.10.4, it is worth mentioning that the analysis of thin shells under flexure with the CB approach presented herein may lead to shear locking, similar to the case described for beam elements in Section 13.14. Selective-reduced integration can be used for the through-thickness shear stiffness terms to address locking issues. All other stiffness terms are calculated with full integration. For the specific case of the four-node CB element shown in Figure 14.25, reduced integration amounts to the use of a single stack of quadrature points (all points in the stack corresponding to $\xi = \eta = 0$), while full integration is conducted using all four stacks shown in the specific figure. ∎

Problems

14.1 We are given the two-shell-element mesh shown in Figure P14.1. The formulation of element 1 is based on the Reissner-Mindlin theory, while element 2 is a discrete Kirchhoff quadrilateral element. Both elements have a thickness of 0.2 m, and are made of linearly elastic, isotropic material with $E = 30$ GPa, $\nu = 0.2$. All translations and rotations of nodes 1 and 2 are fixed to a zero value.



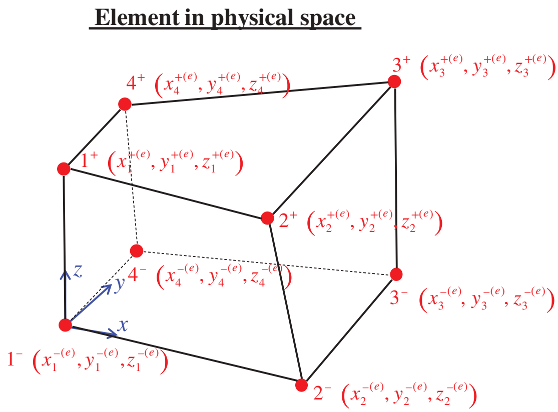
Node	x (m)	y (m)	z (m)
1	0	0	0
2	1	2	–1
3	1	2	4
4	0	0	4
5	5	–1	5
6	4	–1	5

Figure P14.1

a) Obtain the stiffness matrix of each element in the corresponding local coordinate system.
b) Obtain the coordinate transformation array, $[R^{(e)}]$, for each element.

c) Obtain the global stiffness matrix, $[K]$, of the structure.
d) Obtain the nodal displacements and rotations of the model.
e) Calculate the generalized strains and generalized stresses for the quadrature points of each element.

14.2 We are given a continuum-based shell element, for which we have the nodal coordinates of the hexahedral underlying solid element, as shown in Figure P14.2. The material is isotropic, linearly elastic, with $E = 29000$ ksi, $\nu = 0.25$. The nodal coordinates of the shell element are those of the mid-surface.



Node	x (ft)	y (ft)	z (ft)
1^-	0	0	0
2^-	5	0	–0.2
3^-	6	5	–0.1
4^-	1	4	0
1^+	0	0	0.8
2^+	4	–1	0.6
3^+	6	5	1
4^+	–1	3.5	0.9

Figure P14.2

f) Obtain the shell thickness values at each nodal location of the shell element (Note: The thickness is not constant!).
g) Obtain the constraint transformation array, giving the nodal displacements of the solid element in terms of the nodal displacements and rotations in the shell element.
h) Obtain the stiffness matrix of the shell element in the global coordinate system.
i) Obtain the stiffness matrix of the shell element in the global coordinate system, using selective-reduced integration in accordance with Remark 14.10.5.

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